DSA – Feb. 15, 2021

Topics: sinusoidal response, transients, time-constants, transfer functions, pole-zero placement designs, noise reduction, group delay, SMA, EMA, and zero-lag filters, block diagrams, direct, canonical, transposed, and lattice realizations.

$$H(\omega) = H_R(\omega) + jH_I(\omega) = \text{frequency response}$$

$$H(\omega) = |H(\omega)|e^{j\theta(\omega)} = \text{polar form}$$

$$|H(\omega)| = \text{magnitude-response}$$

$$\theta(\omega) = \text{phase-response}$$

$$n_{\text{ph}}(\omega) = -\theta(\omega) = \text{phase-delay}$$

$$n_{\text{gr}}(\omega) = -\frac{d\theta(\omega)}{d\omega} = \text{group-delay}$$

Sinusoidal Response

$$x(n) = e^{j\omega_0 n}, \quad -\infty < n < \infty$$

$$y(n) = \sum_m h(m)x(n-m) = \sum_m h(m)e^{j(n-m)\omega_0} = e^{j\omega_0 n}\sum_m h(m)e^{-j\omega_0 m}$$

$$y(n) = H(\omega_0)e^{j\omega_0 n}$$

$$e^{j\omega_0 n} \xrightarrow{H} H(\omega_0)e^{j\omega_0 n}$$

steady-state sinusoidal response

$$\cos(\omega_0 n) \xrightarrow{H} |H(\omega_0)| \cos\big(\omega_0 n + \arg H(\omega_0)\big)$$

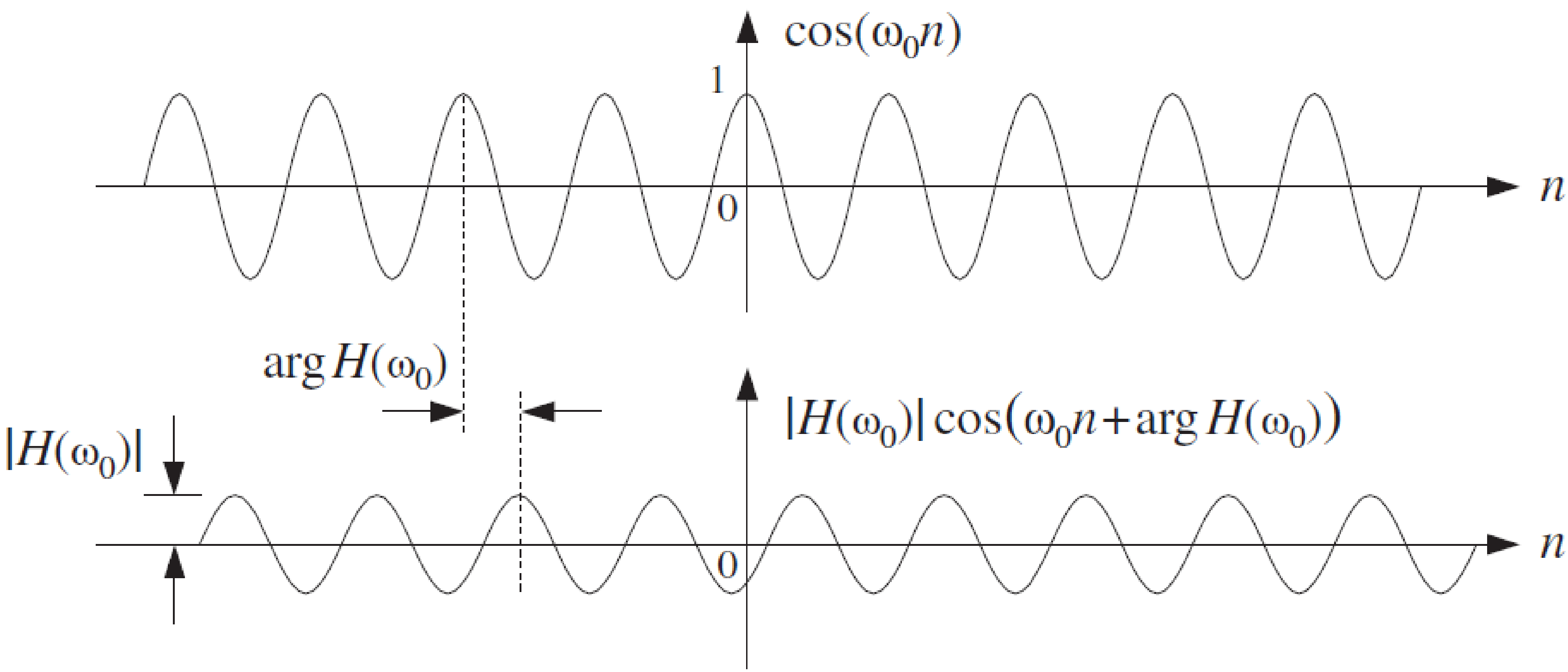
$$\sin(\omega_0 n) \xrightarrow{H} |H(\omega_0)| \sin\big(\omega_0 n + \arg H(\omega_0)\big)$$

Sinusoidal Response

$$\cos(\omega_0 n) \xrightarrow{H} |H(\omega_0)| \cos\left(\omega_0 n + \arg H(\omega_0)\right)$$

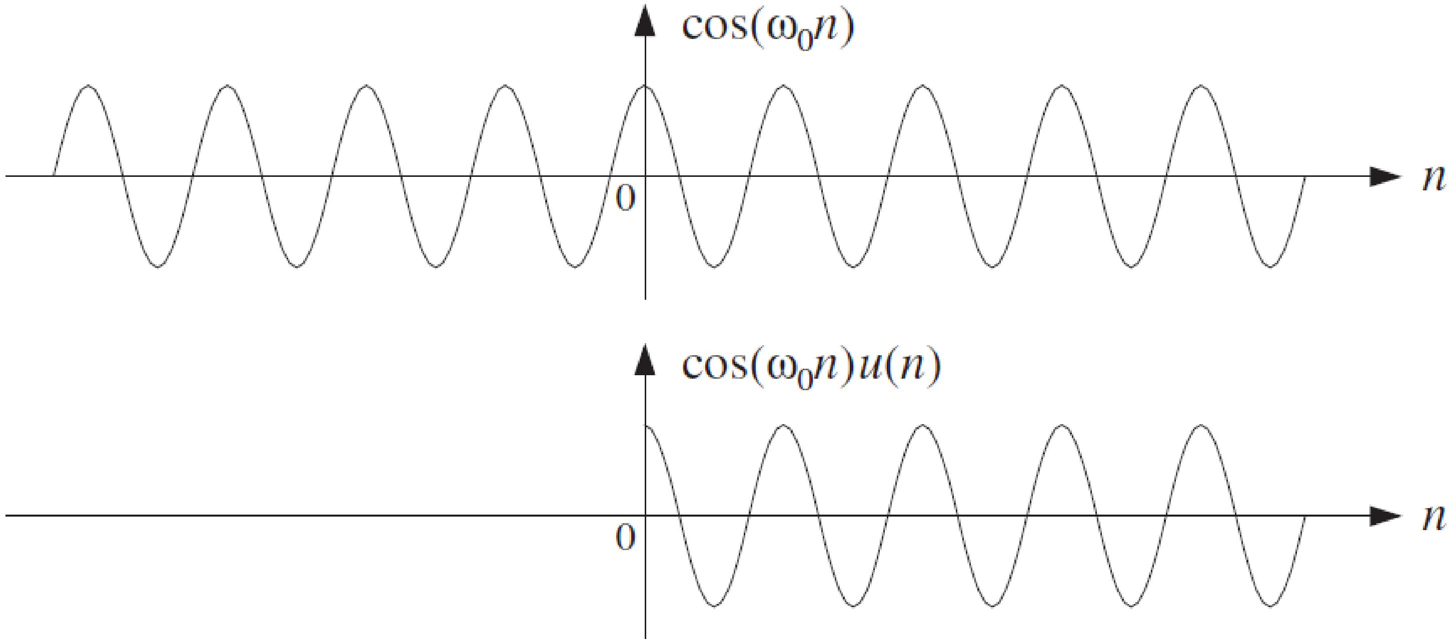
$$\sin(\omega_0 n) \xrightarrow{H} |H(\omega_0)| \sin\left(\omega_0 n + \arg H(\omega_0)\right)$$

steady-state



Sinusoidal Response

transient response



double-sided vs. one-sided sinusoids

transient response

$$x(n) = e^{j\omega_0 n} u(n) \xrightarrow{z} X(z) = \frac{1}{1 - e^{j\omega_0} z^{-1}}$$

$$\boxed{\text{ROC } |z| > |e^{j\omega_0}| = 1}$$

$$H(z) = \frac{N(z)}{D(z)} = \frac{N(z)}{(1 - p_1 z^{-1})(1 - p_2 z^{-1}) \cdots (1 - p_M z^{-1})}$$

$$\boxed{|p_i| < 1}$$

stable and causal $H(z)$ with distinct poles strictly inside the UC, and deg$N < M$+1

$$Y(z) = H(z)X(z) = \frac{N(z)}{(1 - e^{j\omega_0} z^{-1})(1 - p_1 z^{-1})(1 - p_2 z^{-1}) \cdots (1 - p_M z^{-1})}$$

$$Y(z) = \frac{C}{1 - e^{j\omega_0} z^{-1}} + \frac{B_1}{1 - p_1 z^{-1}} + \frac{B_2}{1 - p_2 z^{-1}} + \cdots + \frac{B_M}{1 - p_M z^{-1}}$$

$$\boxed{C = H(z)\big|_{z=e^{j\omega_0}} = H(\omega_0)}$$

transient response

$$C = (1 - e^{j\omega_0} z^{-1}) Y(z) \big|_{z=e^{j\omega_0}} = \left[\cancel{(1 - e^{j\omega_0} z^{-1})} \frac{H(z)}{\cancel{1 - e^{j\omega_0} z^{-1}}} \right]_{z=e^{j\omega_0}}$$

$$\boxed{C = H(z) \big|_{z=e^{j\omega_0}} = H(\omega_0)}$$

$$Y(z) = \frac{H(\omega_0)}{1 - e^{j\omega_0} z^{-1}} + \frac{B_1}{1 - p_1 z^{-1}} + \frac{B_2}{1 - p_2 z^{-1}} + \cdots + \frac{B_M}{1 - p_M z^{-1}}$$

$$\boxed{y(n) = H(\omega_0) e^{j\omega_0 n} + B_1 p_1^n + B_2 p_2^n + \cdots + B_M p_M^n} \quad n \geq 0$$

steady-state

transients

$$|p_i| < 1$$

$$y(n) \to H(\omega_0) e^{j\omega_0 n} \quad \text{as} \quad n \to \infty$$

transient response

$$\boxed{y(n) = H(\omega_0)e^{j\omega_0 n} + B_1 p_1^n + B_2 p_2^n + \cdots + B_M p_M^n} \quad n \geq 0; \qquad |p_i| < 1$$

$$\rho = \max_i |p_i|$$

pole of largest radius, or, lying closest to the UC from the inside

$$\rho^{n_{\text{eff}}} = \epsilon$$

$$\boxed{n_{\text{eff}} = \frac{\ln \epsilon}{\ln \rho} = \frac{\ln(1/\epsilon)}{\ln(1/\rho)}}$$

time constant in samples

$$\epsilon = 1\% = 0.01$$

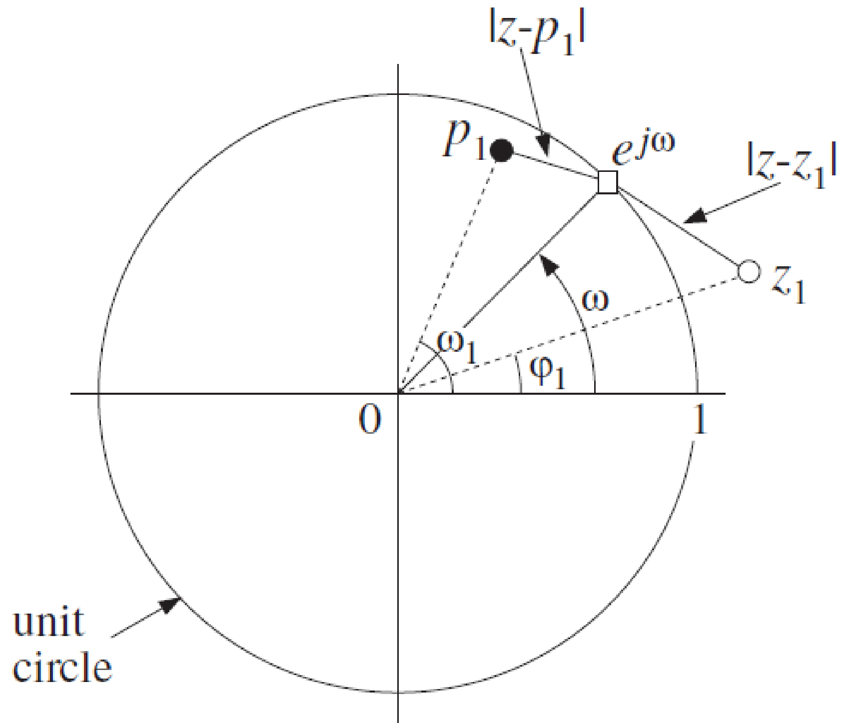
$$\epsilon = 0.1\% = 10^{-3}$$

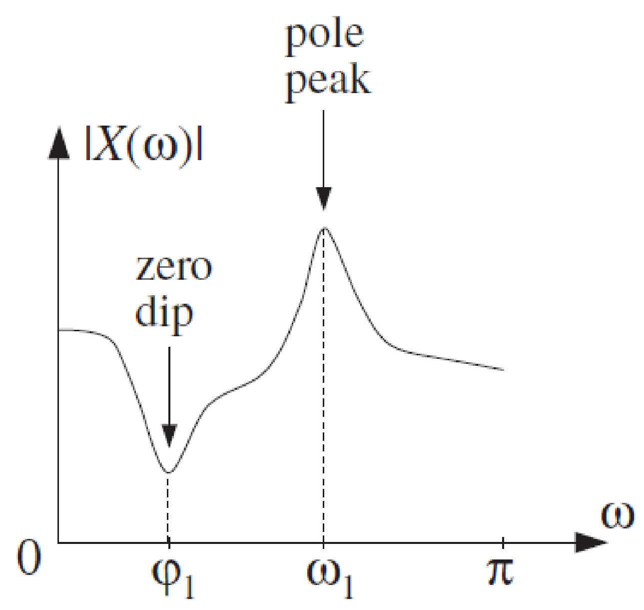
40-dB and 60-dB time constants

Pole-Zero Pattern

$$X(z) = \frac{1 - z_1 z^{-1}}{1 - p_1 z^{-1}} = \frac{z - z_1}{z - p_1}$$

$$X(\omega) = \frac{e^{j\omega} - z_1}{e^{j\omega} - p_1} \quad \Rightarrow \quad |X(\omega)| = \frac{|e^{j\omega} - z_1|}{|e^{j\omega} - p_1|}$$



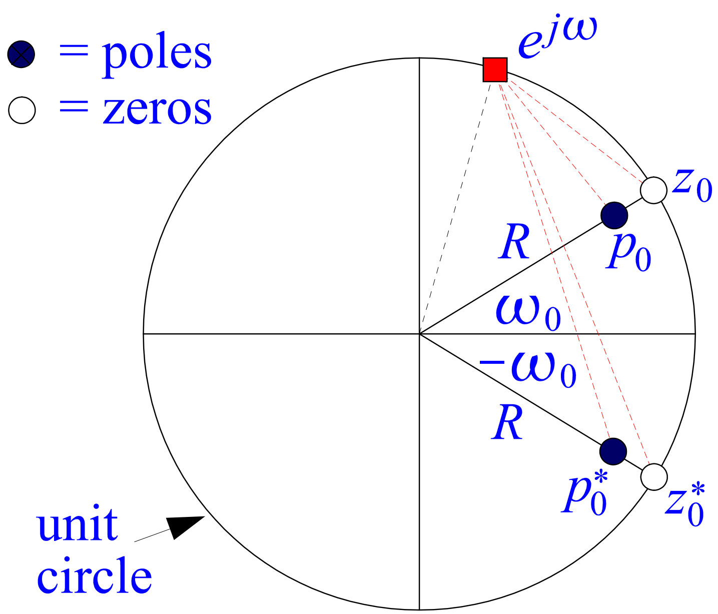


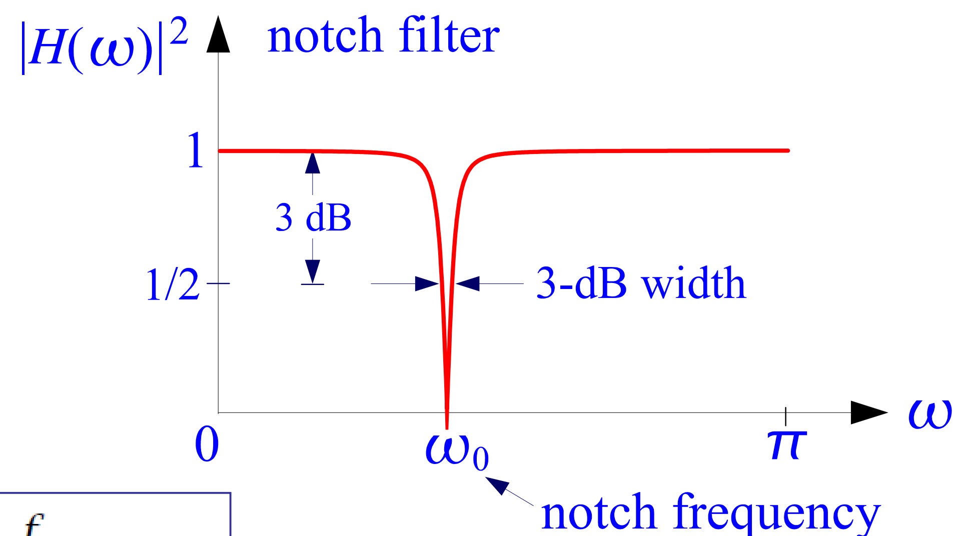
Filter Design by Pole-Zero Placement

see the files
notch-digital.pdf
notch-analog.pdf

$$z_0 = e^{j\omega_0}$$
$$p_0 = Re^{j\omega_0}$$

$$|H(\omega)| = G \cdot \frac{|e^{j\omega} - z_0|}{|e^{j\omega} - p_0|} \cdot \frac{|e^{j\omega} - z_0^*|}{|e^{j\omega} - p_0^*|}$$





$$\Delta f = \frac{f_s}{\pi}(1 - R)$$

project-2 topics

s21pr02.pdf

1. group delay property
2. noise reduction
3. signal enhancement
4. transients, time constants
5. SMA, EMA, PMA, DEMA filters

group delay property for narrow-band pulses

$$H(\omega) = H_R(\omega) + j\,H_I(\omega) = \text{frequency response}$$

$$H(\omega) = |H(\omega)|\,e^{j\theta(\omega)} = \text{polar form}$$

$$|H(\omega)| = \text{magnitude-response}$$

$$\theta(\omega) = \text{phase-response}$$

$$n_{\text{ph}}(\omega) = -\theta(\omega) = \text{phase-delay}$$

$$n_{\text{gr}}(\omega) = -\frac{d\theta(\omega)}{d\omega} = \text{group-delay}$$

group delay property for narrow-band pulses

DTFT properties – O&S Ch.2 Table 2-2

$$p(n) \overset{\mathrm{FT}}{\longleftrightarrow} P(\omega)$$

$$p(n-n_0)) \overset{\mathrm{FT}}{\longleftrightarrow} e^{-j\omega n_0}P(\omega)$$

$$e^{j\omega_0 n}p(n) \overset{\mathrm{FT}}{\longleftrightarrow} P(\omega-\omega_0)$$

$$e^{j\omega_0 n}p(n-n_0) \overset{\mathrm{FT}}{\longleftrightarrow} e^{-j(\omega-\omega_0)n_0}P(\omega-\omega_0)$$

$$x(n) = e^{j\omega_0 n}p(n) \overset{\mathrm{FT}}{\longleftrightarrow} X(\omega) = P(\omega-\omega_0)$$

$$y(n) = h(n) * x(n) \overset{\mathrm{FT}}{\longleftrightarrow} Y(\omega) = H(\omega)X(\omega) = H(\omega)P(\omega-\omega_0)$$

group delay property for narrow-band pulses

$$Y(\omega) = H(\omega)P(\omega-\omega_0) = |H(\omega)|\,e^{j\theta(\omega)}P(\omega-\omega_0)$$

$$Y(\omega) = H(\omega)P(\omega-\omega_0) \approx |H(\omega_0)|\,e^{j\theta(\omega_0)+j(\omega-\omega_0)\theta'(\omega_0)}P(\omega-\omega_0)$$

$$|H_0| = |H(\omega_0)|\,, \quad \theta_0 = \theta(\omega_0)\,, \quad n_0 = -\theta'(\omega_0)$$

$$Y(\omega) = H(\omega)P(\omega-\omega_0) \approx |H_0|\,e^{j\theta_0}\,e^{-j(\omega-\omega_0)n_0}P(\omega-\omega_0)$$

$$y(n) \approx |H_0|\,e^{j\theta_0}\,e^{j\omega_0 n}p(n-n_0) = |H_0|\,e^{j(\omega_0 n+\theta_0)}p(n-n_0)$$

phase delay

group delay

Transfer Function Realizations

direct form, DF-1 realization

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

$$Y(z) = H(z)X(z) = \left[\frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}\right] X(z)$$

$$(1 + a_1 z^{-1} + a_2 z^{-2})Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2})X(z)$$

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

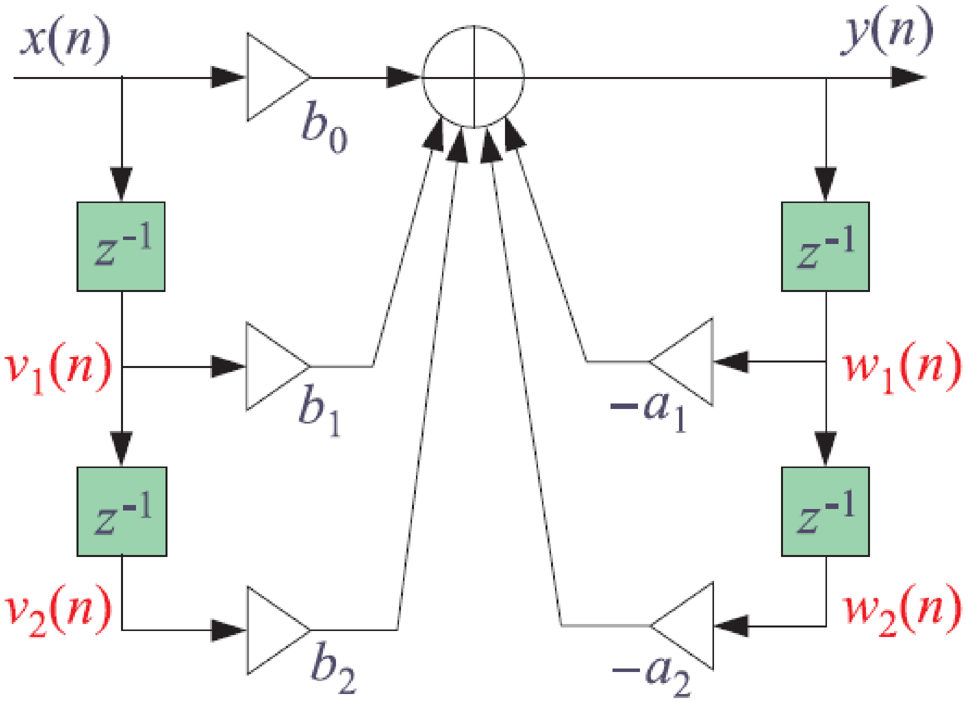
$$y(n) = \underbrace{-a_1 y(n-1) - a_2 y(n-2)}_{\text{feeedback recursive part}} + \underbrace{b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)}_{\text{feed-forward non-recursive part}}$$

Transfer Function Realizations

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

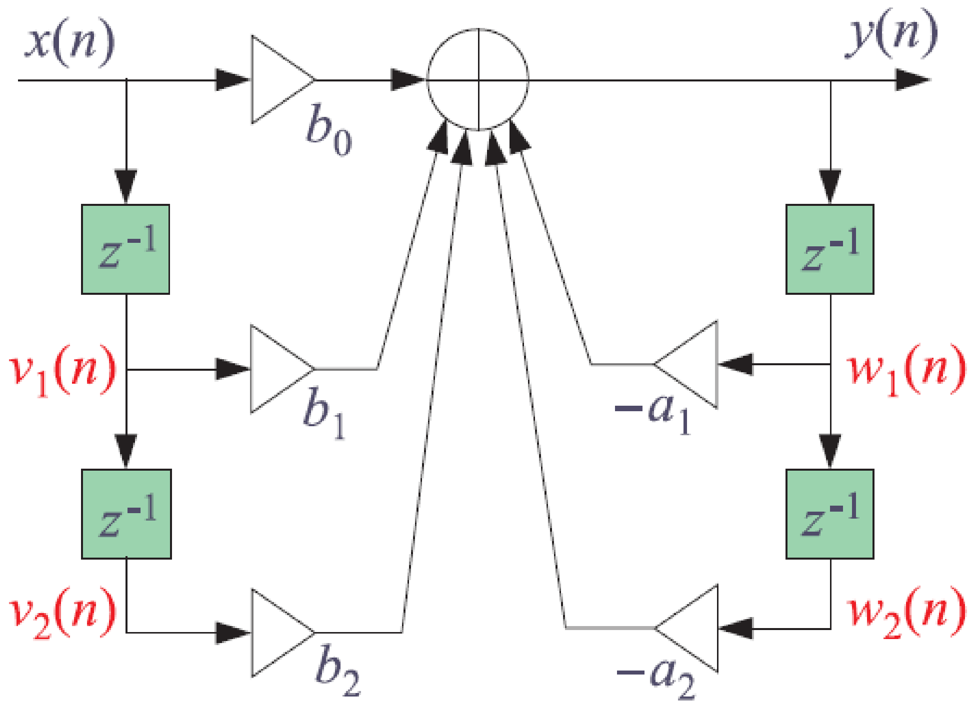
direct form, DF-1 realization

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$



$$\begin{aligned} v_1(n) &= x(n-1) \\ v_2(n) &= x(n-2) = v_1(n-1) \\ w_1(n) &= y(n-1) \\ w_2(n) &= y(n-2) = w_1(n-1) \end{aligned}$$

$$\begin{aligned} v_1(n+1) &= x(n) \\ v_2(n+1) &= v_1(n) \\ w_1(n+1) &= y(n) \\ w_2(n+1) &= w_1(n) \end{aligned}$$



direct form, DF-1 realization

sample processing

initialize w_1, w_2, v_1, v_2

for each input sample x, do,

$$y = -a_1 w_1 - a_2 w_2 + b_0 x + b_1 v_1 + b_2 v_2$$

$$w_2 = w_1$$

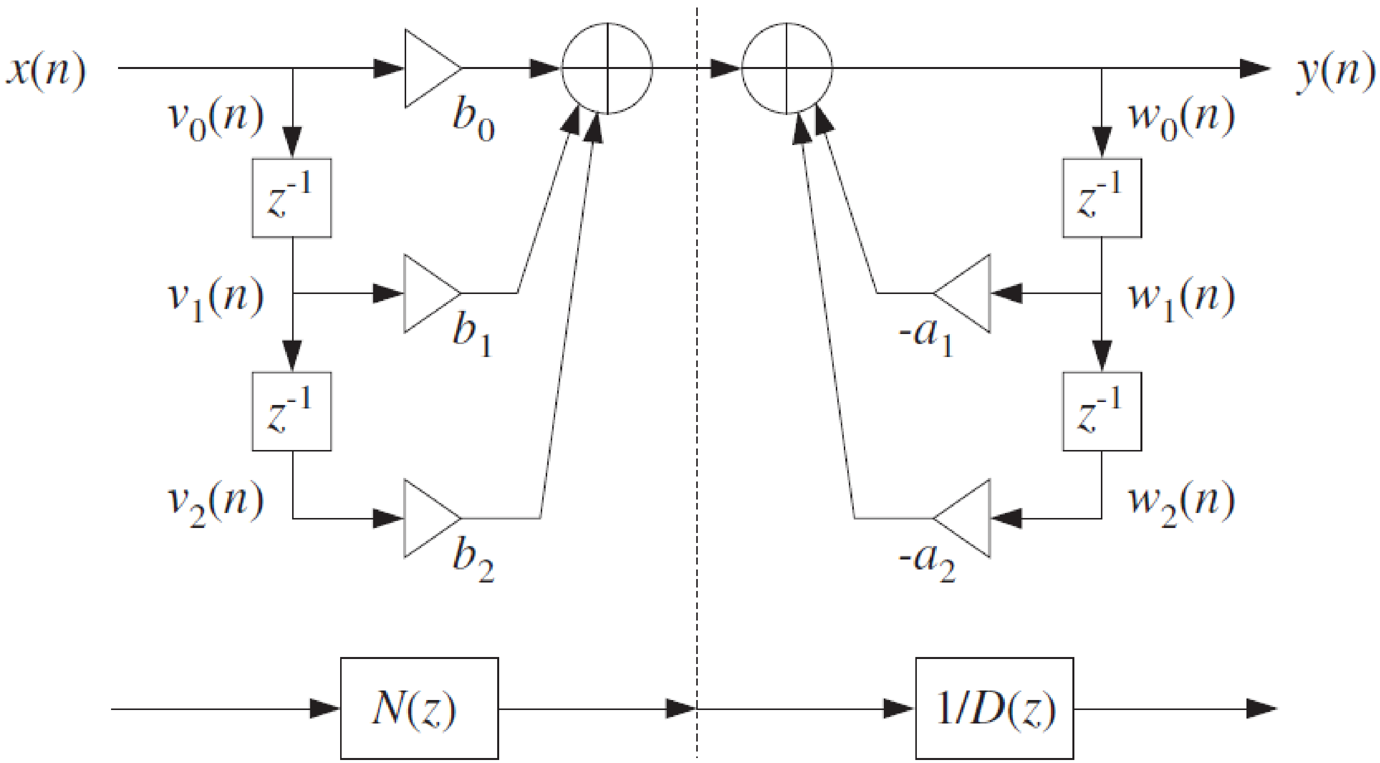
$$w_1 = y$$

$$v_2 = v_1$$

$$v_1 = x$$

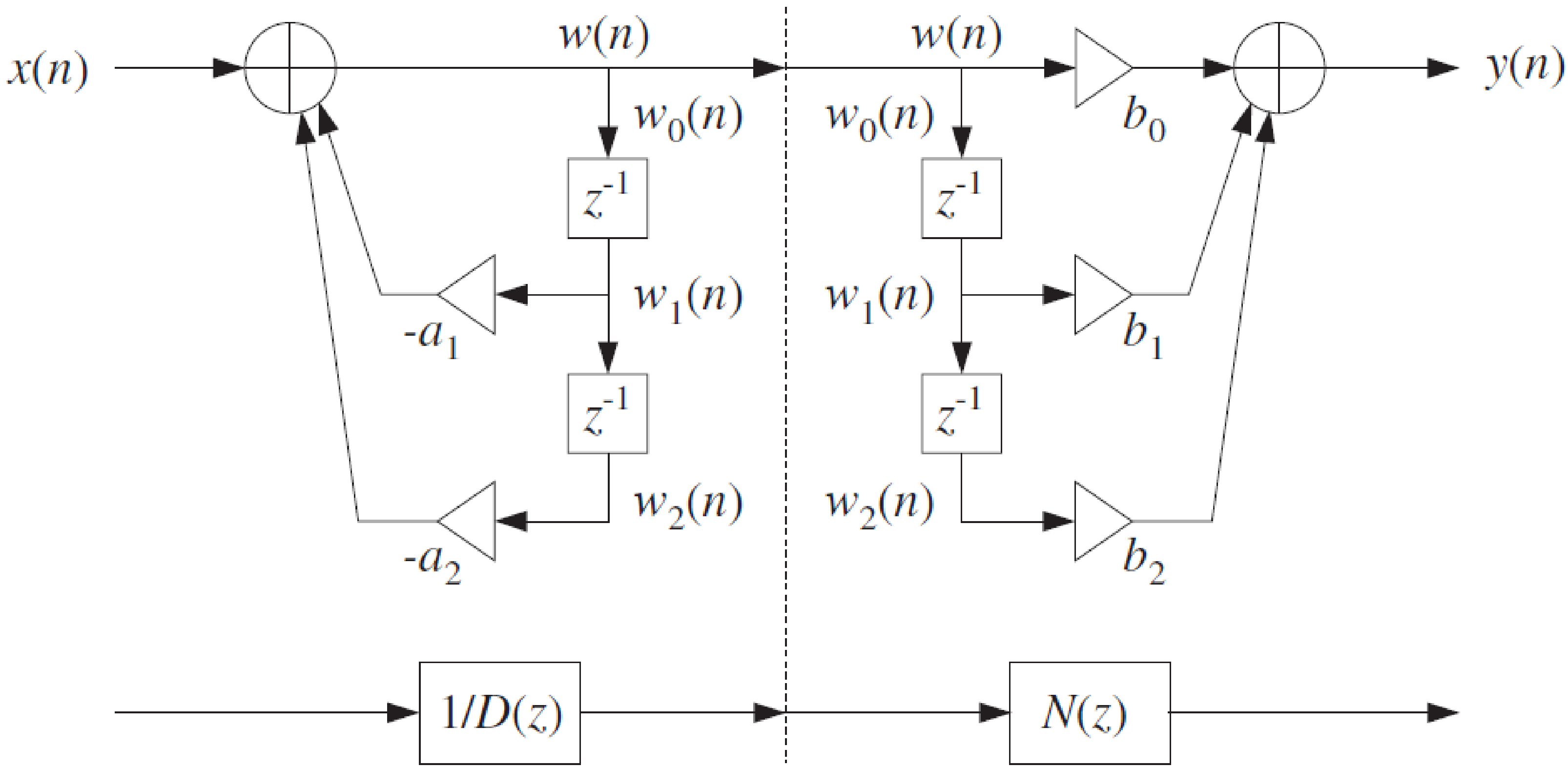
canonical form, DF-2 realization

$$y(n) = \underbrace{-a_1 y(n-1) - a_2 y(n-2)}_{\text{feeedback recursive part}} + \underbrace{b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)}_{\text{feed-forward non-recursive part}}$$



$$H(z) = N(z) \cdot \frac{1}{D(z)} = \frac{1}{D(z)} \cdot N(z)$$

canonical form, DF-2 realization

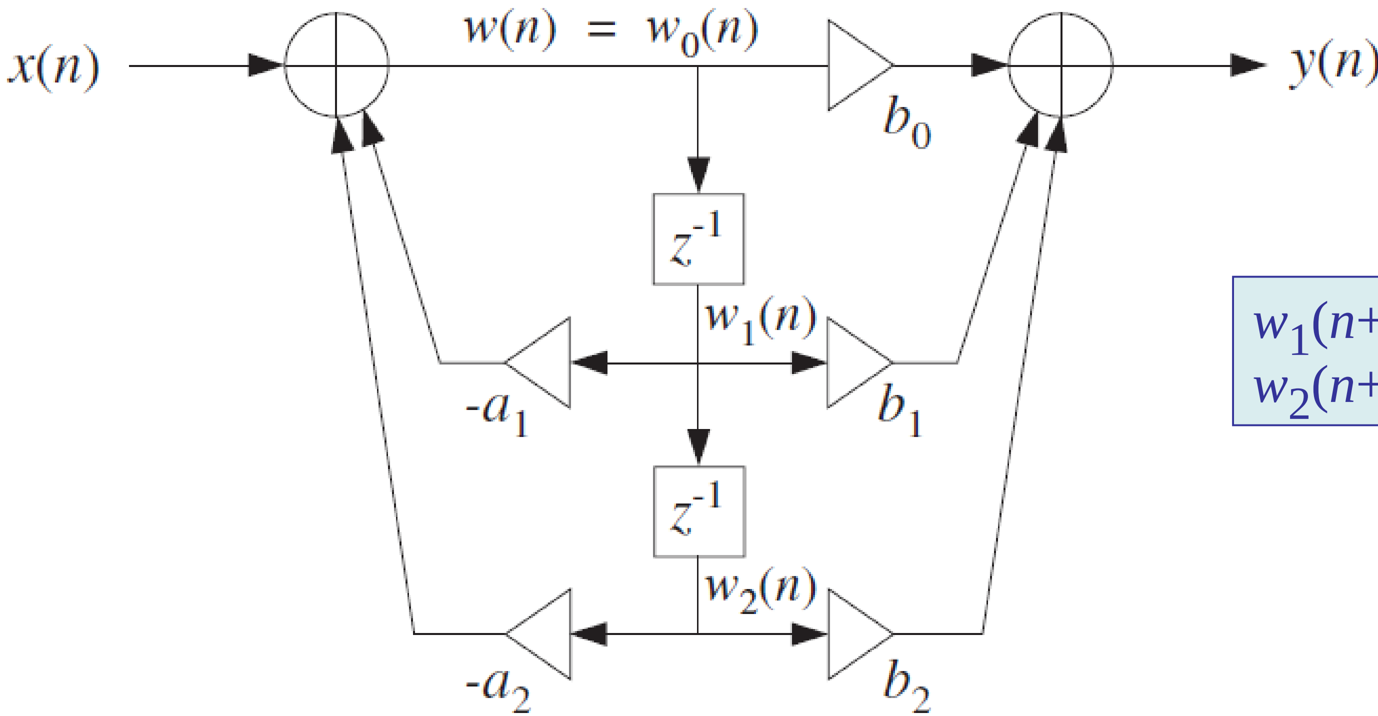


$$W(z) = \frac{1}{D(z)} X(z)$$

$$Y(z) = N(z)W(z) = N(z) \cdot \frac{1}{D(z)} X(z) = H(z)X(z)$$

$$H(z) = N(z)\cdot\frac{1}{D(z)} = \frac{1}{D(z)}\cdot N(z)$$

canonical form, DF-2 realization

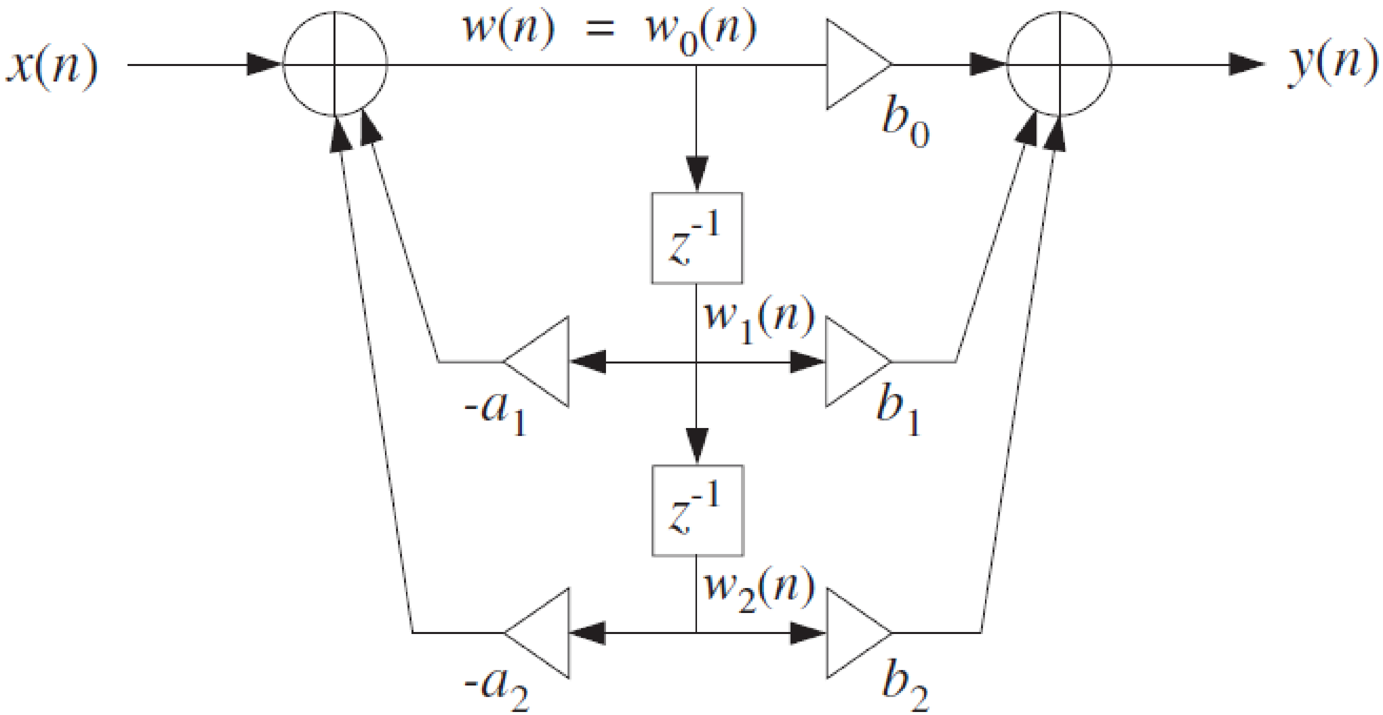


$$w_1(n+1) = w_0(n)$$
$$w_2(n+1) = w_1(n)$$

$$W(z) = \frac{1}{D(z)}X(z) \quad \Rightarrow \quad (1 + a_1 z^{-1} + a_2 z^{-2})W(z) = X(z)$$

$$Y(z) = N(z)W(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2})W(z)$$

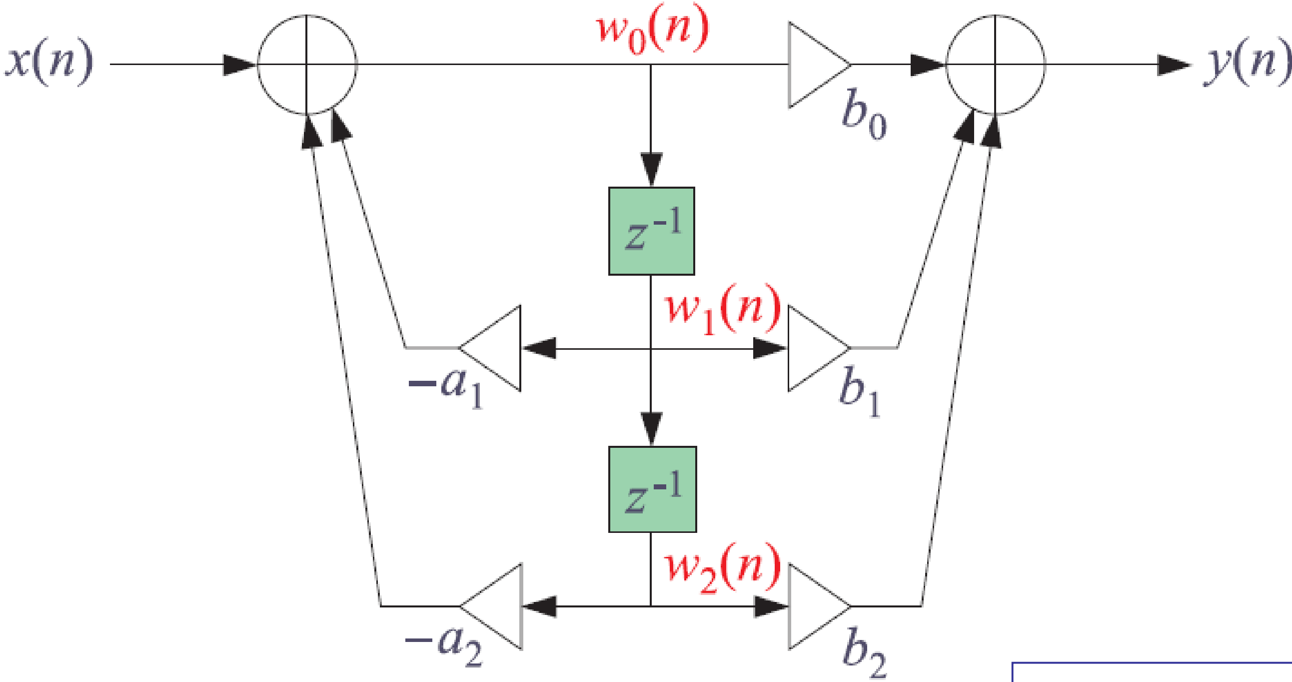
canonical form, DF-2 realization



$$w(n) = -a_1 w(n-1) - a_2 w(n-2) + x(n)$$

$$y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2)$$

canonical form, DF-2 realization



$$w_2(n+1) = w_1(n)$$
$$w_1(n+1) = w_0(n)$$

sample processing

initialize w_1, w_2

for each input sample x, do,

$$w_0 = -a_1 w_1 - a_2 w_2 + x$$
$$y = b_0 w_0 + b_1 w_1 + b_2 w_2$$
$$w_2 = w_1$$
$$w_1 = w_0$$

transposed realization

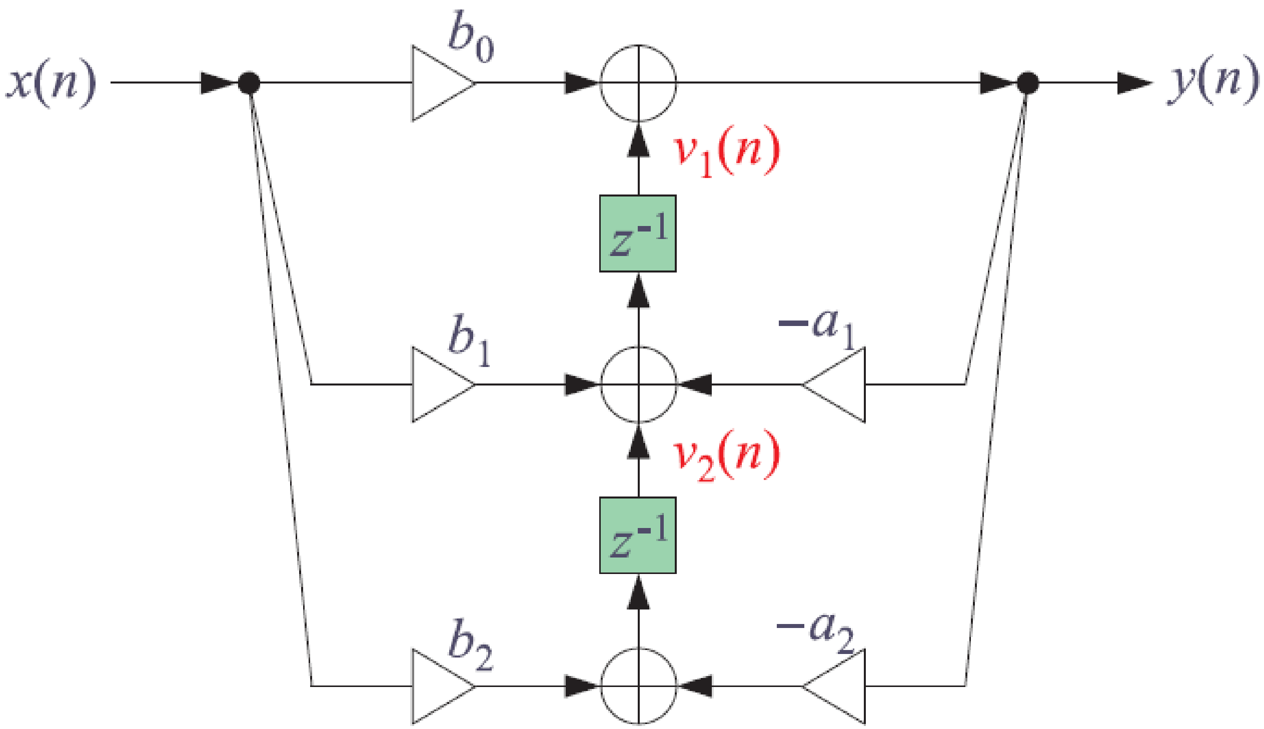
$$Y(z) = H(z)X(z) = \left[\frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}\right] X(z)$$

$$(1 + a_1 z^{-1} + a_2 z^{-2})Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2})X(z)$$

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

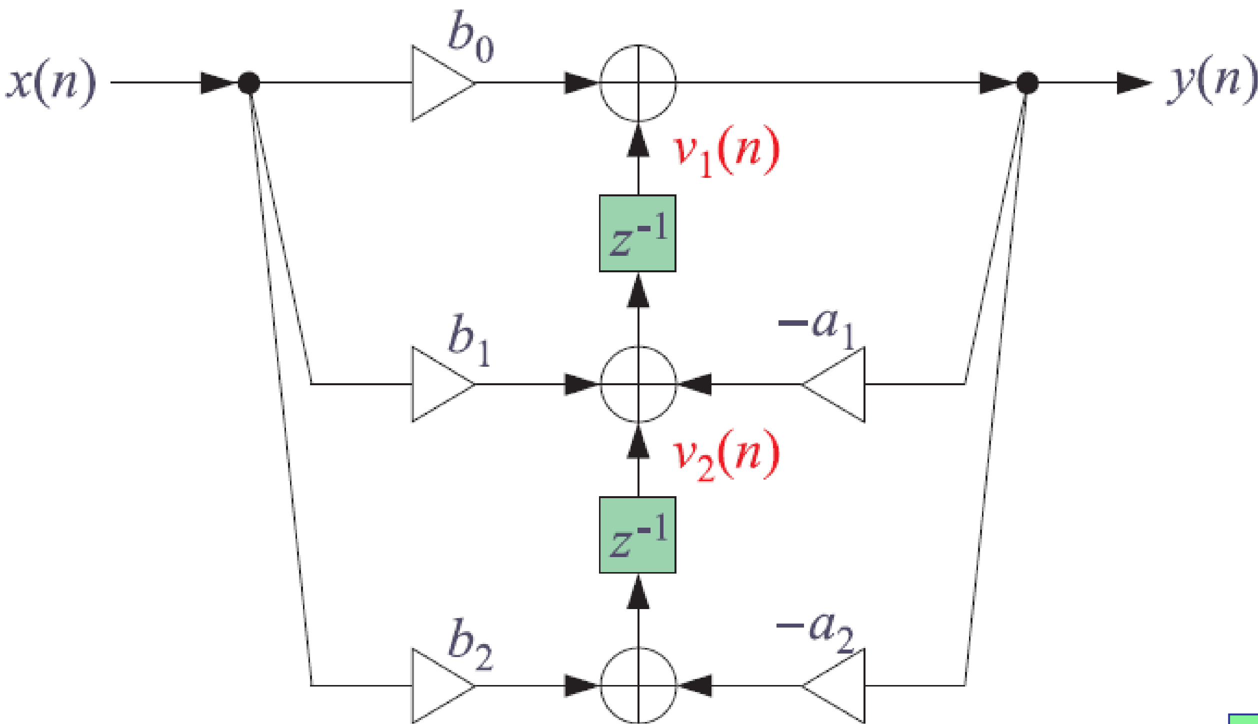
$$Y(z) = b_0 X(z) + z^{-1}\left[b_1 X(z) - a_1 Y(z) + z^{-1}\big(b_2 X(z) - a_2 Y(z)\big)\right]$$

transposed realization



$$Y(z) = b_0 X(z) + \underbrace{z^{-1}\Big[b_1 X(z) - a_1 Y(z) + \underbrace{z^{-1}\big(b_2 X(z) - a_2 Y(z)\big)\Big]}_{V_2(z)}}_{V_1(z)}$$

transposed realization



sample processing

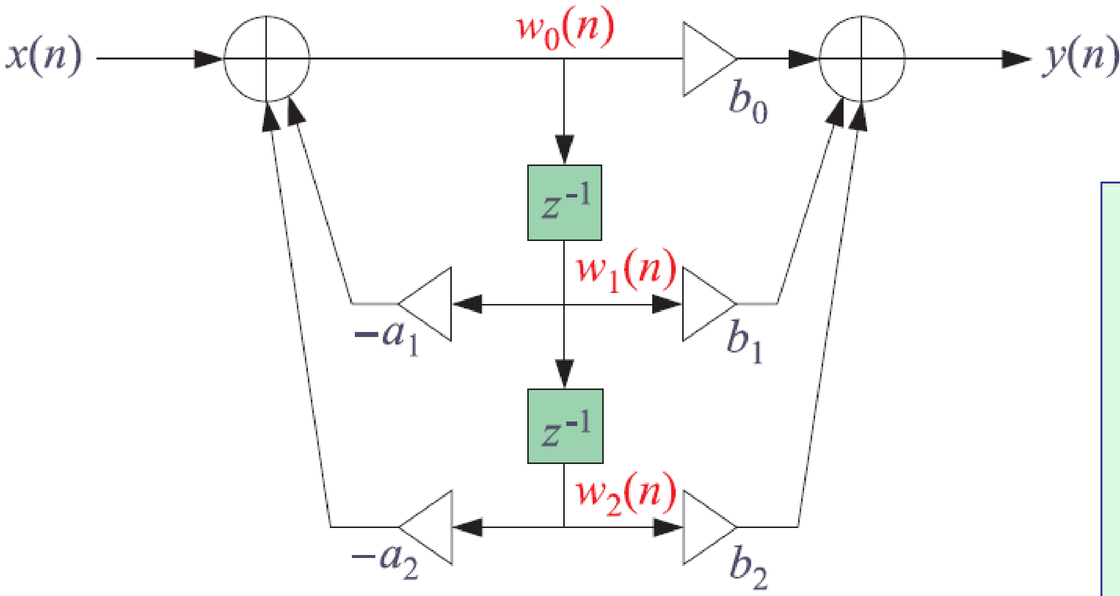
initialize v_1, v_2

for each input sample x, do,

$$y = b_0 x + v_1$$

$$v_1 = b_1 x - a_1 y + v_2$$

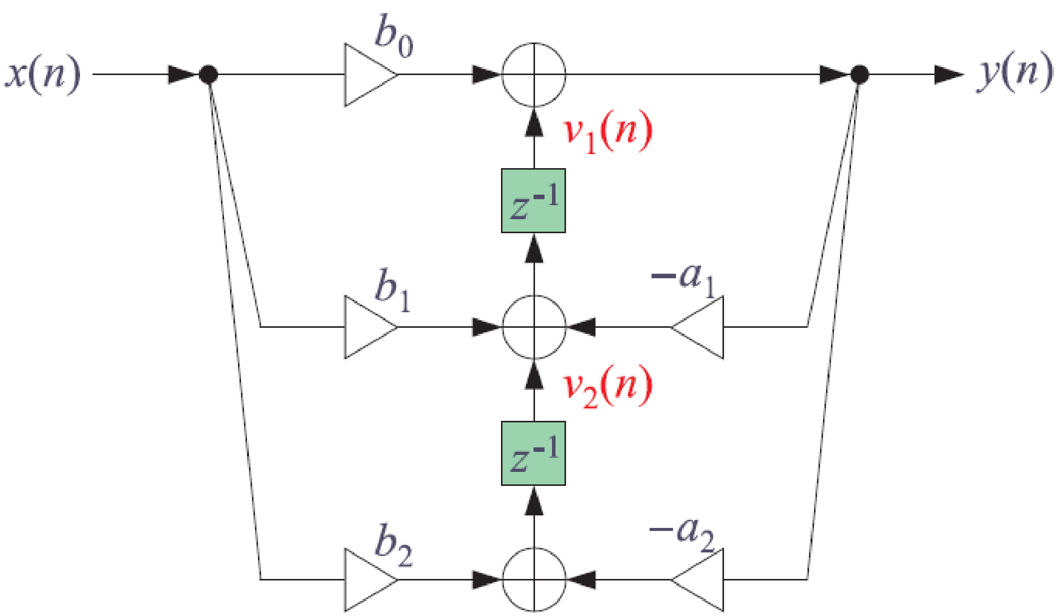
$$v_2 = b_2 x - a_2 y$$



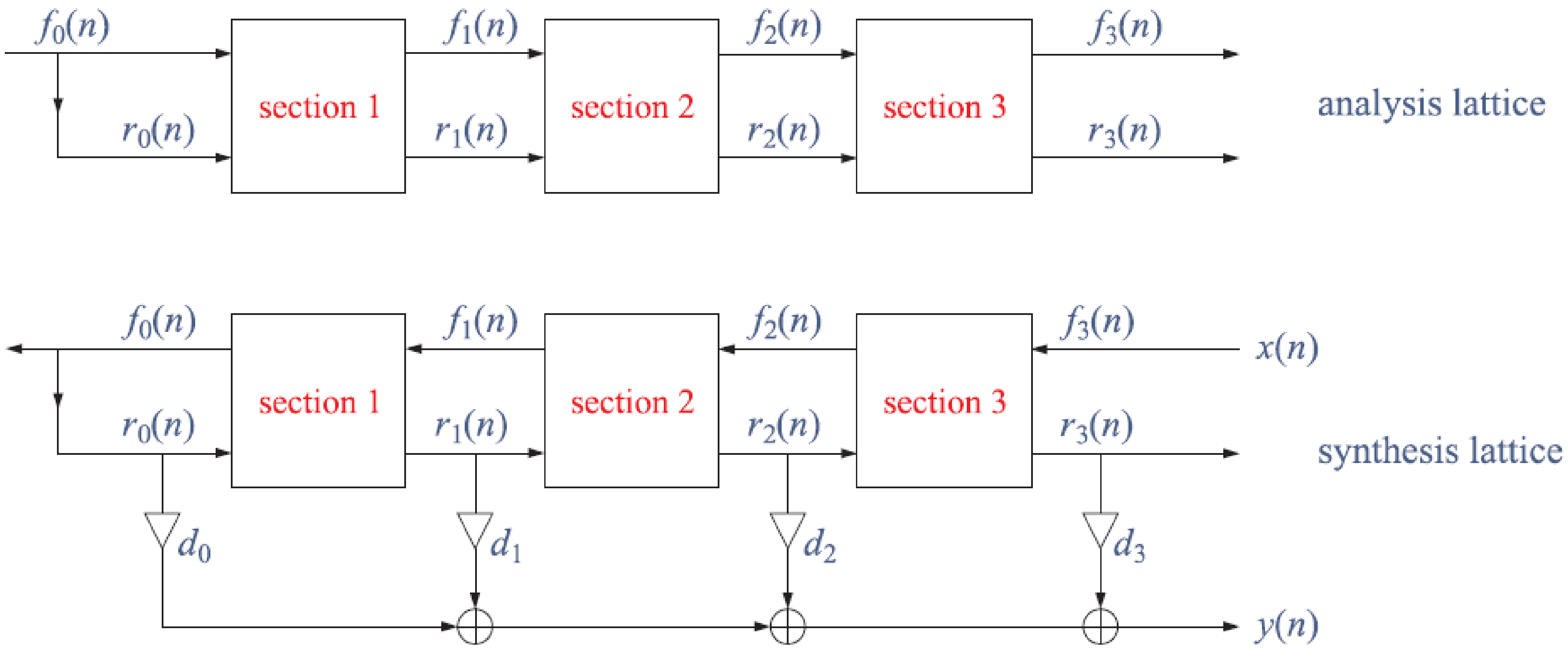
transposed realization

transposition rules

1. replace adders by nodes
2. replace nodes by adders
3. reverse all flows
4. interchange input and output



lattice / ladder realizations



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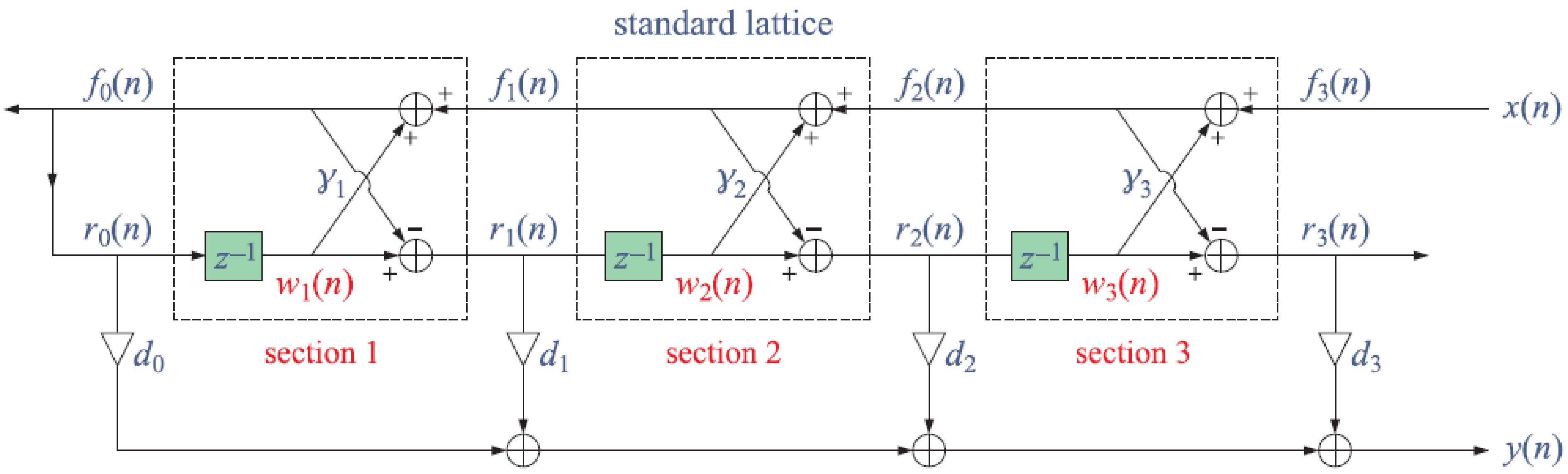
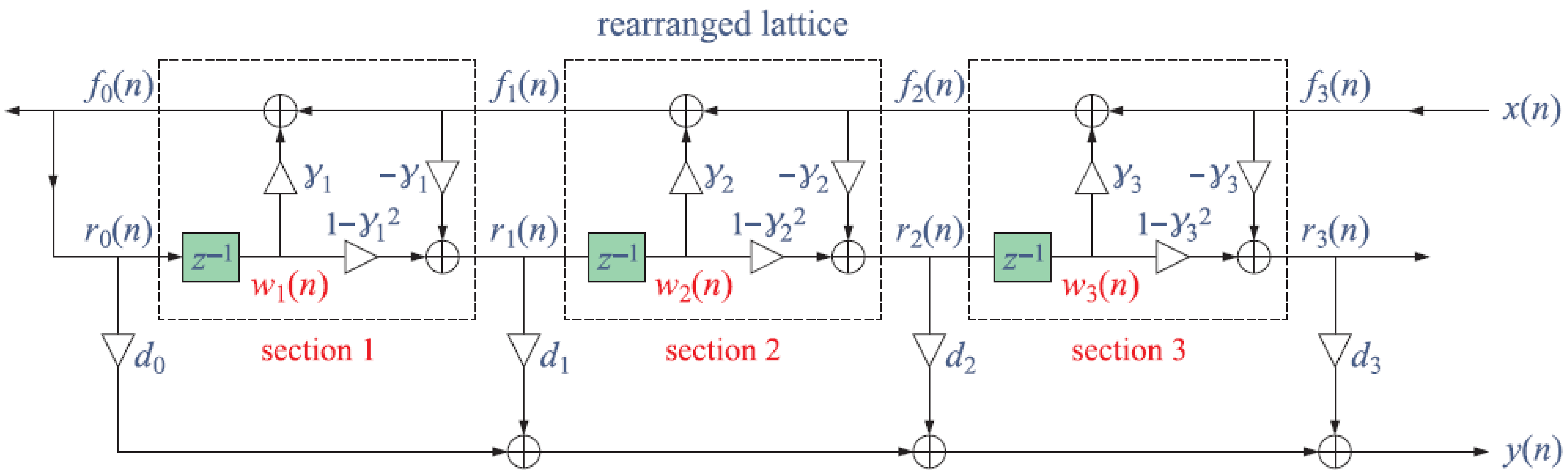
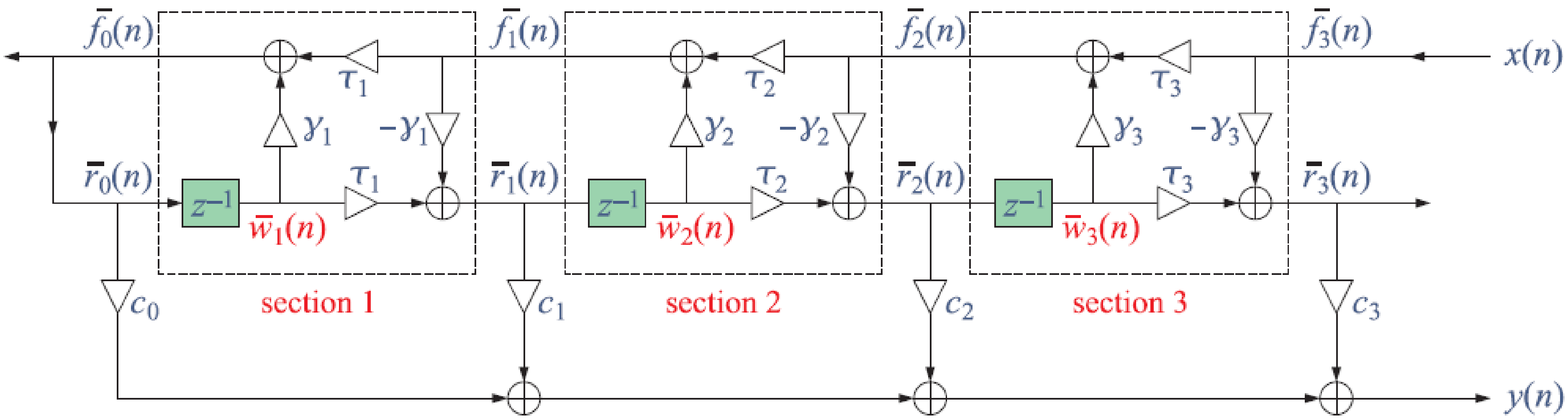


lattice / ladder realizations



lattice / ladder realizations

normalized lattice



$$\tau_p = (1 - \gamma_p^2)^{1/2}$$