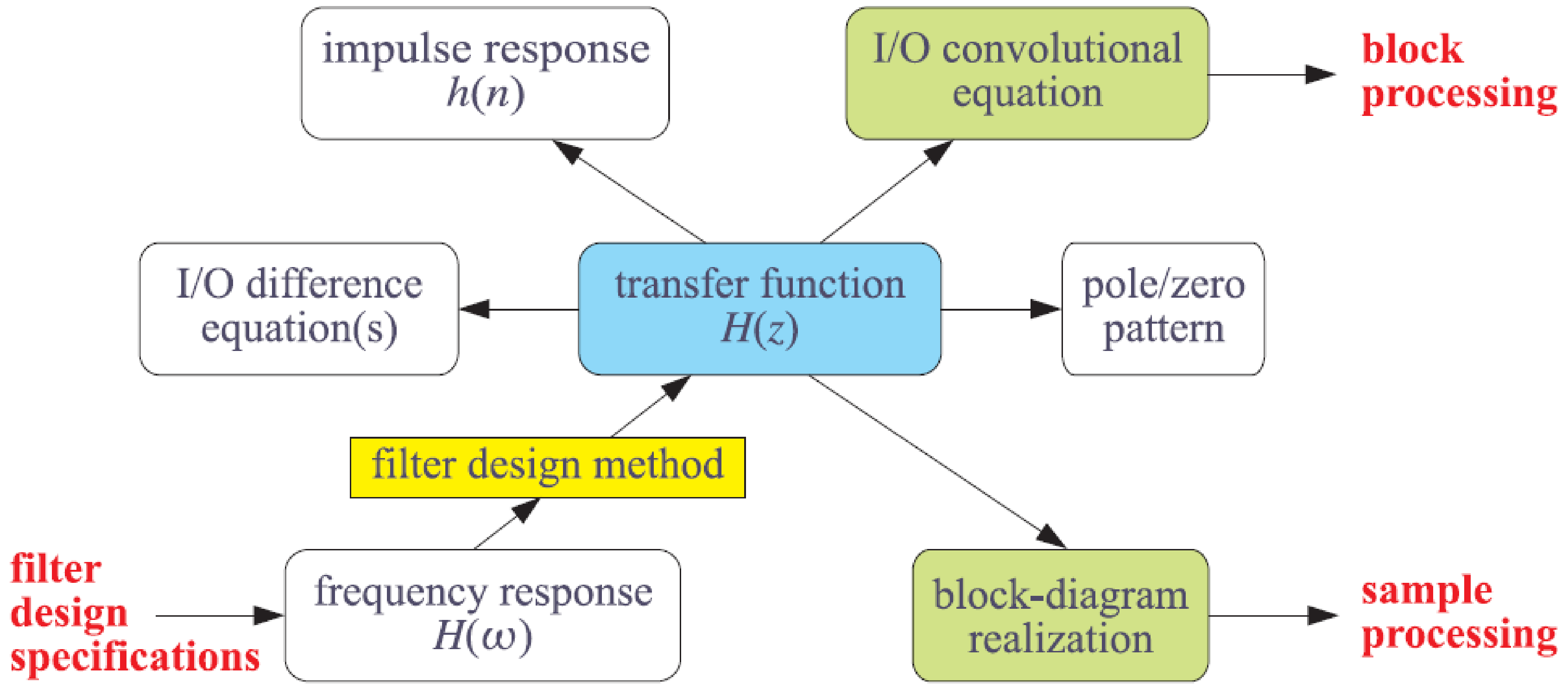
DSA – Feb. 8, 2021

Topics: LTI systems, convolution, overlap-add method, z-transforms, ROC, inverse z-transforms, stability vs. causality, transfer functions, block diagrams, steady-state sinusoidal response, transients, pole-zero placement designs.



Continuous-Time LTI Systems

$x(t)$ input $\longrightarrow$ $\boxed{\begin{array}{c}\text{LTI system}\\ h(t),\ H(s)\end{array}}$ $\longrightarrow$ $y(t)$ output

LTI systems are convolvers

$$\boxed{y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau}$$

direct form

LTI form

Discrete-Time LTI Systems

$x[n] \longrightarrow$ LTI system $h[n]$, $H(z)$ $\longrightarrow y[n]$

input sequence

output sequence

LTI systems are convolvers

$$\boxed{y(n) = \sum_{m=-\infty}^{\infty} h(m)\, x(n-m) = \sum_{m=-\infty}^{\infty} x(m)\, h(n-m)}$$

direct form

LTI form

Discrete-Time LTI Systems

I2SP – Ch.3
O&S – Ch.2

$$x(n) \longrightarrow \boxed{\begin{array}{c}\text{LTI system}\\ h(n),\ H(z)\end{array}} \longrightarrow y(n)$$

input — output

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\,z^{-n} = \text{transfer function, } z\text{-transform}$$

$$H(\omega) = H(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n)\,e^{-j\omega n} = \text{frequency response, DTFT}$$

$$h(n) = \int_{-\pi}^{\pi} H(\omega)e^{j\omega n}\,\frac{d\omega}{2\pi} = \text{inverse DTFT}$$

$$\boxed{\omega = \frac{2\pi f}{f_s}\,, \qquad z = e^{j\omega}}$$

$$h(n) = \oint_{\text{u.c.}} H(z)\,z^{n}\,\frac{dz}{2\pi j z} = \text{contour integral}$$

Discrete-Time LTI Systems

$$H(\omega) = H_R(\omega) + jH_I(\omega) = \text{frequency response}$$

$$H(\omega) = |H(\omega)|e^{j\theta(\omega)} = \text{polar form}$$

$$|H(\omega)| = \text{magnitude-response}$$

$$\theta(\omega) = \text{phase-response}$$

$$n_{\mathrm{ph}}(\omega) = -\theta(\omega) = \text{phase-delay}$$

$$n_{\mathrm{gr}}(\omega) = -\frac{d\theta(\omega)}{d\omega} = \text{group-delay}$$

Discrete-Time LTI Systems

$$\text{if } h(n) = \text{real-valued},$$

$$H(-\omega) = H(\omega)^* = \text{Hermitian property}$$

$$H_R(-\omega) = H_R(\omega) = \text{even in } \omega$$

$$H_I(-\omega) = -H_I(\omega) = \text{odd in } \omega$$

$$|H(-\omega)| = |H(\omega)| = \text{even in } \omega$$

$$\theta(-\omega) = -\theta(\omega) = \text{odd in } \omega$$

$$n_{\text{ph}}(\omega) = \text{odd in } \omega$$

$$n_{\text{gr}}(\omega) = \text{even in } \omega$$

Sinusoidal Response

Input complex sinusoid is assumed to be infinitely-long and double-sided (i.e., we are looking at the steady-state behavior of the system),

$$Ae^{j\omega n} \longrightarrow \boxed{H(\omega)} \longrightarrow H(\omega)Ae^{j\omega n}$$

Moreover, if the LTI system has real-valued impulse response $h(n)$, then the following results also hold for real-valued input sinusoids,

$$\mathrm{Re}\left[(Ae^{j\omega n}\right] \longrightarrow \boxed{H(\omega)} \longrightarrow \mathrm{Re}\left[H(\omega)Ae^{j\omega n}\right]$$

$$\mathrm{Im}\left[(Ae^{j\omega n}\right] \longrightarrow \boxed{H(\omega)} \longrightarrow \mathrm{Im}\left[H(\omega)Ae^{j\omega n}\right]$$

complex sinusoids can be viewed as the eigenfunctions of LTI systems

Filtering in the Frequency Domain

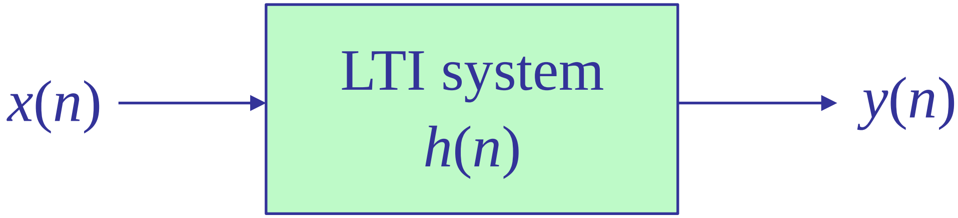
$$x(n) \longrightarrow \boxed{H(\omega)} \longrightarrow y(n)$$

$$x(n) = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} \frac{d\omega}{2\pi}$$

$$y(n) = \int_{-\pi}^{\pi} H(\omega)\, X(\omega)\, e^{j\omega n} \frac{d\omega}{2\pi}$$

i.e., filtering is spectral reshaping of the input spectrum

Discrete-Time Convolution



$$y(n) = \sum_{m=-\infty}^{\infty} h(m)\, x(n-m) = \sum_{m=-\infty}^{\infty} x(m)\, h(n-m)$$

Convolution Computation

$$y_n = \sum_m x_m h_{n-m} = \text{LTI form}$$

$$= \sum_m h_m x_{n-m} = \text{direct form}$$

$$= \sum_{\substack{i,j \\ i+j=n}} h_i x_j = \text{convolution table form}$$

table forms

matrix forms

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & 0 & 0 \\ h_1 & h_0 & 0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 & 0 \\ h_3 & h_2 & h_1 & h_0 & 0 \\ 0 & h_3 & h_2 & h_1 & h_0 \\ 0 & 0 & h_3 & h_2 & h_1 \\ 0 & 0 & 0 & h_3 & h_2 \\ 0 & 0 & 0 & 0 & h_3 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ x_1 & x_0 & 0 & 0 \\ x_2 & x_1 & x_0 & 0 \\ x_3 & x_2 & x_1 & x_0 \\ x_4 & x_3 & x_2 & x_1 \\ 0 & x_4 & x_3 & x_2 \\ 0 & 0 & x_4 & x_3 \\ 0 & 0 & 0 & x_4 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

conv, convmtx in MATLAB

Convolution Computation

	h_0	h_1	h_2	h_3	0	0	0	0
x_0	x_0h_0	x_0h_1	x_0h_2	x_0h_3	0	0	0	0
x_1	0	x_1h_0	x_1h_1	x_1h_2	x_1h_3	0	0	0
x_2	0	0	x_2h_0	x_2h_1	x_2h_2	x_2h_3	0	0
x_3	0	0	0	x_3h_0	x_3h_1	x_3h_2	x_3h_3	0
x_4	0	0	0	0	x_4h_0	x_4h_1	x_4h_2	x_4h_3
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

LTI table

add vertically

time $\longrightarrow$ n

Convolution Computation

	x_0	x_1	x_2	x_3	x_4	0	0	0
h_0	h_0x_0	h_0x_1	h_0x_2	h_0x_3	h_0x_4	0	0	0
h_1	0	h_1x_0	h_1x_1	h_1x_2	h_1x_3	h_1x_4	0	0
h_2	0	0	h_2x_0	h_2x_1	h_2x_2	h_2x_3	h_2x_4	0
h_3	0	0	0	h_3x_0	h_3x_1	h_3x_2	h_3x_3	h_3x_4
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

direct-form table

$\downarrow$

add vertically

time $\longrightarrow$ n

Convolution Computation

$\longrightarrow j$

$i \downarrow$

	x_0	x_1	x_2	x_3	x_4
h_0	h_0x_0	h_0x_1	h_0x_2	h_0x_3	h_0x_4
h_1	h_1x_0	h_1x_1	h_1x_2	h_1x_3	h_1x_4
h_2	h_2x_0	h_2x_1	h_2x_2	h_2x_3	h_2x_4
h_3	h_3x_0	h_3x_1	h_3x_2	h_3x_3	h_3x_4

convolution table

Convolution Computation - Example

$$\mathbf{h} = [1,\ -2,\ 0,\ 3]$$

$$\mathbf{x} = [4,\ 3,\ 2,\ 1,\ 2]$$

$$\Rightarrow \quad \mathbf{y} = \mathbf{h} * \mathbf{x} = [4,\ -5,\ -4,\ 9,\ 9,\ 2,\ 3,\ 6]$$

`y = conv(h,x)` – MATLAB code

4	-8	0	12				
	3	-6	0	9			
		2	-4	0	6		
			1	-2	0	3	
				2	-4	0	6
4	-5	-4	9	9	2	3	6

LTI table

Convolution Computation - Example

direct-form table

4	3	2	1	2			
	-8	-6	-4	-2	-4		
		0	0	0	0	0	
			12	9	6	3	6
4	-5	-4	9	9	2	3	6

convolution table

	4	3	2	1	2			
1	4	3	2	1	2			
-2	-8	-6	-4	-2	-4			
0	0	0	0	0	0			
3	12	9	6	3	6			
	4	-5	-4	9	9	2	3	6

Convolution Computation - Example

matrix forms

$$\begin{bmatrix} 4 \\ -5 \\ -4 \\ 9 \\ 9 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 3 & 0 & -2 & 1 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 0 & 0 & 3 & 0 & -2 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}$$

```
h = [1,-2, 0, 3]';              % column vector
x = [4, 3, 2, 1, 2]';           % column vector

H = convmtx(h,length(x));       % convolution matrix
X = convmtx(x,length(h));       % convolution matrix

y = H * x;                      % equivalent to conv(h,x)
y = X * h;                      % also equivalent to conv(h,x)
```

Numerical Evaluation of CT Convolution

$$y(t) = \int_{-\infty}^{\infty} h(t')x(t-t')dt' = \int_{-\infty}^{\infty} h(t-t')x(t')dt'$$

$$y(t) = \int h(t-\tau)x(\tau)d\tau = \lim_{T\to 0}\left[T\sum_m h(t-t_m)x(t_m)\right]$$

$$t_n = nT$$

$$t_m = mT$$

$$y(t) = \int h(t-\tau)x(\tau)d\tau \approx T\sum_m h(t-t_m)x(t_m)$$

$$\boxed{y(t_n) = \int h(t_n-\tau)x(\tau)d\tau \approx T\sum_m h(t_n-t_m)x(t_m)}$$

```
y = T * conv(h,x);
```

Convolution Computation

```
% ---------------------- DIY version of CONV ----------------

function y = myconv(h,x)

M = length(h)-1;                   % filter order
L = length(x);                     % input length
y = zeros(size(x));                % inherit column/row nature of x
                                   % but final y length is L+M

h = h(:);                          % make h,x into columns
x = x(:);

for n=0:L-1+M,
   m = max(0,n-L+1):min(n,M);      % vector index
   y(n+1) = h(m+1).' * x(n-m+1);   % dot product
end

% -----------------------------------------------------------
```

Convolution Computation

$$\mathbf{h} = \boxed{\quad M+1 \quad}$$

$$\mathbf{x} = \boxed{\qquad\quad L \qquad\quad}$$

$$\mathbf{y} = \mathbf{h} * \mathbf{x} = \boxed{\qquad\quad L \qquad\quad \vdots \quad M \quad}$$

$$\mathbf{h} = [h_0, h_1, \ldots, h_M] \quad = \text{filter of order } M$$

$$\mathbf{x} = [x_0, x_1, \ldots, x_{L-1}] \quad = \text{input of length } L$$

$$\mathbf{y} = [y_0, y_1, \ldots, y_{L-1+M}] \quad = \text{output of length } L + M$$

$$L_y = L_x + L_h - 1$$

Convolution Computation

$$y(n) = \sum_{m} h(m)x(n-m)$$

$$0 \le m \le M$$

$$0 \le n - m \le L - 1$$

$$\boxed{0 \le n \le L - 1 + M}$$

range of n

$$\xrightarrow{\qquad}$$

$$0 \le m \le M$$

$$n - L + 1 \le m \le n$$

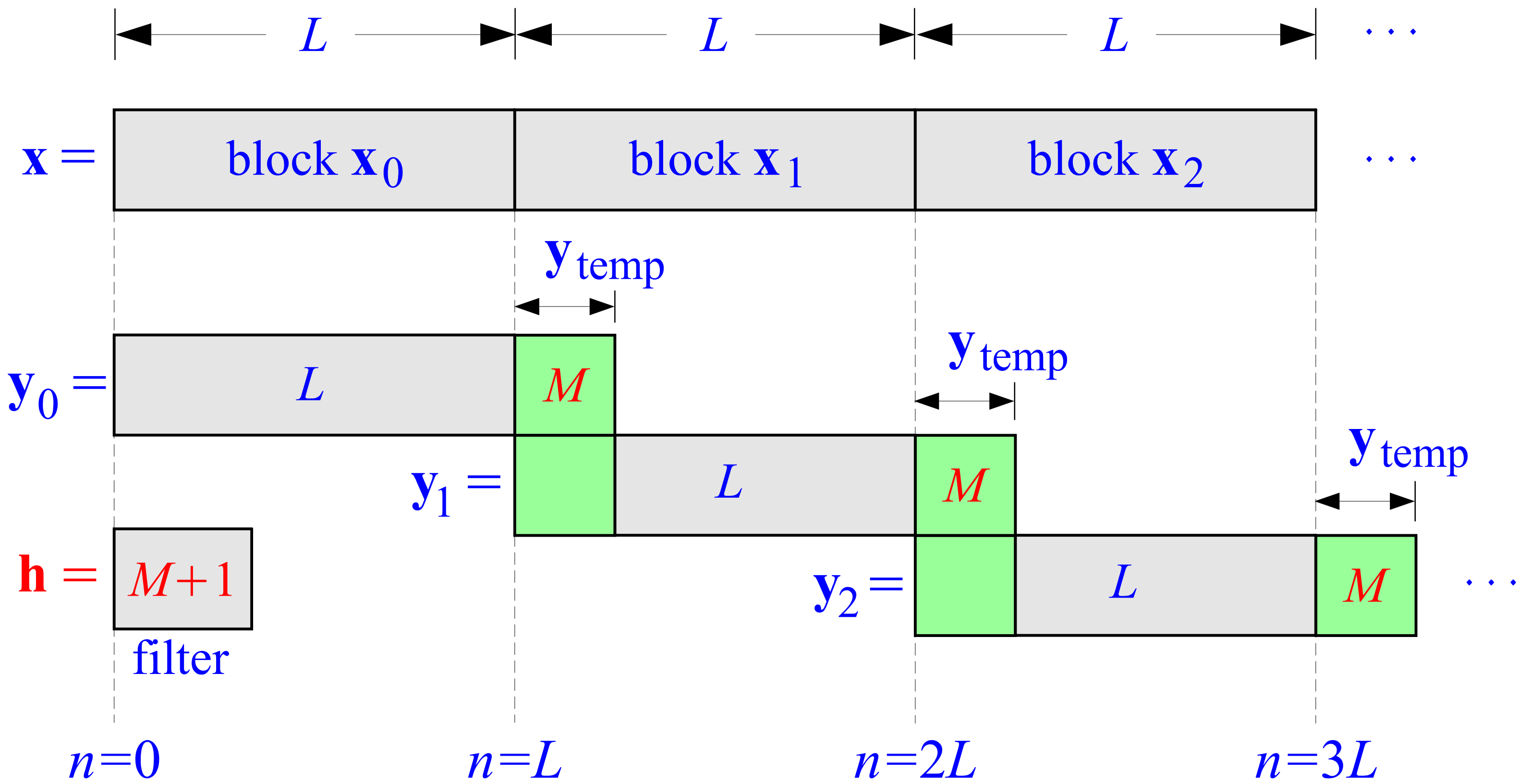
$$\boxed{\max(0, n - L + 1) \le m \le \min(n, M)}$$

range of m

$$y(n) = \sum_{m=\max(0,n-L+1)}^{\min(n,M)} h(m)x(n-m)$$

$$n = 0, 1, \ldots, L + M - 1$$

Overlap-Add Block Convolution Method (time domain)

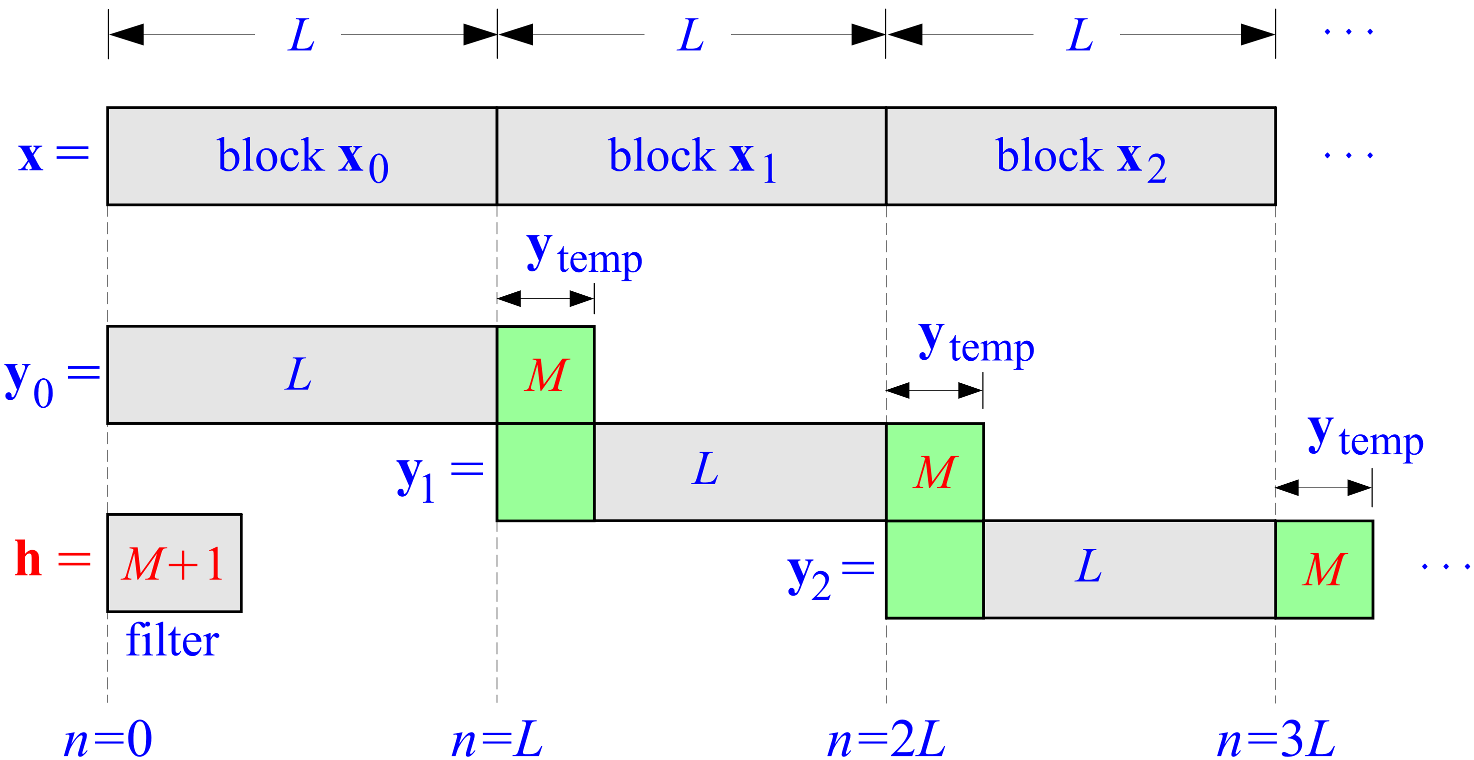


$$\mathbf{y}_0 = \mathbf{h} * \mathbf{x}_0$$

$$\mathbf{y}_1 = \mathbf{h} * \mathbf{x}_1$$

$$\mathbf{y}_2 = \mathbf{h} * \mathbf{x}_2$$

Overlap-Add Block Convolution Method (time domain)



for each length-L input block $\mathbf{x}$ *do:*

1. *compute length-*$(L+M)$ *output:* $\mathbf{y} = \mathbf{h} * \mathbf{x}$
2. *for* $i = 0, 1, \ldots, M-1$:

   $y(i) = y(i) + y_{\text{temp}}(i)$ (overlap)

   $y_{\text{temp}}(i) = y(i+L)$ (save tail)
3. *for* $i = 0, 1, \ldots, L-1$:

   output $y(i)$

z-transforms

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$ (z-transform)

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$ (transfer function)

basic properties

see, **sztable.pdf**, for more

$$a_1 x_1(n) + a_2 x_2(n) \xrightarrow{z} a_1 X_1(z) + a_2 X_2(z)$$ (linearity)

$$x(n) \xrightarrow{z} X(z) \quad \Rightarrow \quad x(n-D) \xrightarrow{z} z^{-D} X(z)$$ (delay)

$$y(n) = h(n) * x(n) \quad \Rightarrow \quad Y(z) = H(z) X(z)$$ (convolution)

region of convergence (ROC)

$$\text{Region of Convergence} = \left\{ z \in \mathbb{C} \;\middle|\; X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \neq \infty \right\}$$

complex z-plane

finite and infinite geometric series

$$1 + x + \cdots + x^{N-1} = \frac{1 - x^N}{1 - x}$$

for $|x| < 1$

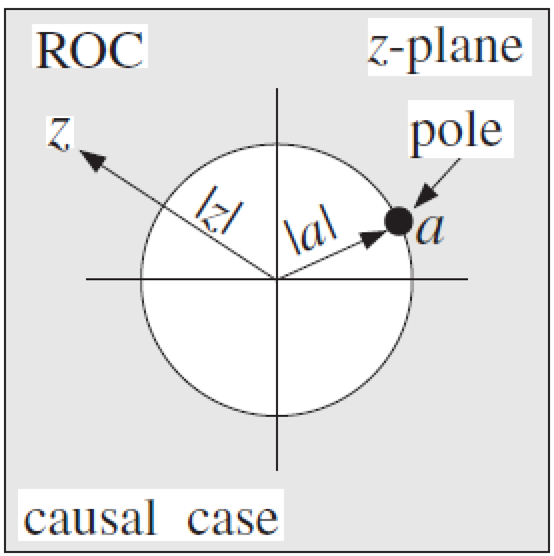
$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}$$

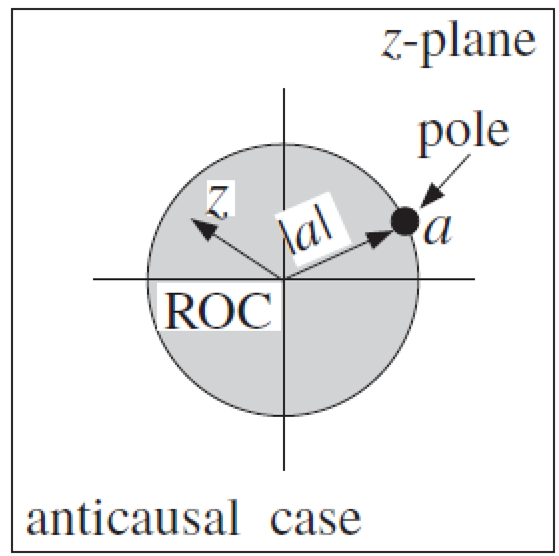
$$x + x^2 + x^3 + \cdots = \frac{x}{1 - x}$$

basic z-transform pair

$$a^n u(n) \xrightarrow{z} \frac{1}{1 - az^{-1}}, \qquad \text{with } |z| > |a|$$

$$-a^n u(-n-1) \xrightarrow{z} \frac{1}{1 - az^{-1}}, \qquad \text{with } |z| < |a|$$





basic z-transform pair

$$\frac{1}{1 - az^{-1}} \longleftrightarrow \begin{cases} a^n u(n)\,, & \text{ROC, } |z| > |a| \\ -a^n u(-n-1)\,, & \text{ROC, } |z| < |a| \end{cases}$$

	stable	unstable
causal	$\|a\| < 1$	$\|a\| > 1$
anticausal	$\|a\| > 1$	$\|a\| < 1$

z-transform pairs

$f(n)$	$F(z)$
$\delta(n-D)$	z^{-D}
$u(n)$	$\dfrac{1}{1-z^{-1}}$
$nu(n)$	$\dfrac{z^{-1}}{(1-z^{-1})^2}$
$n^2u(n)$	$\dfrac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
$a^nu(n)$	$\dfrac{1}{1-az^{-1}}$
$na^nu(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$

Z-transform properties

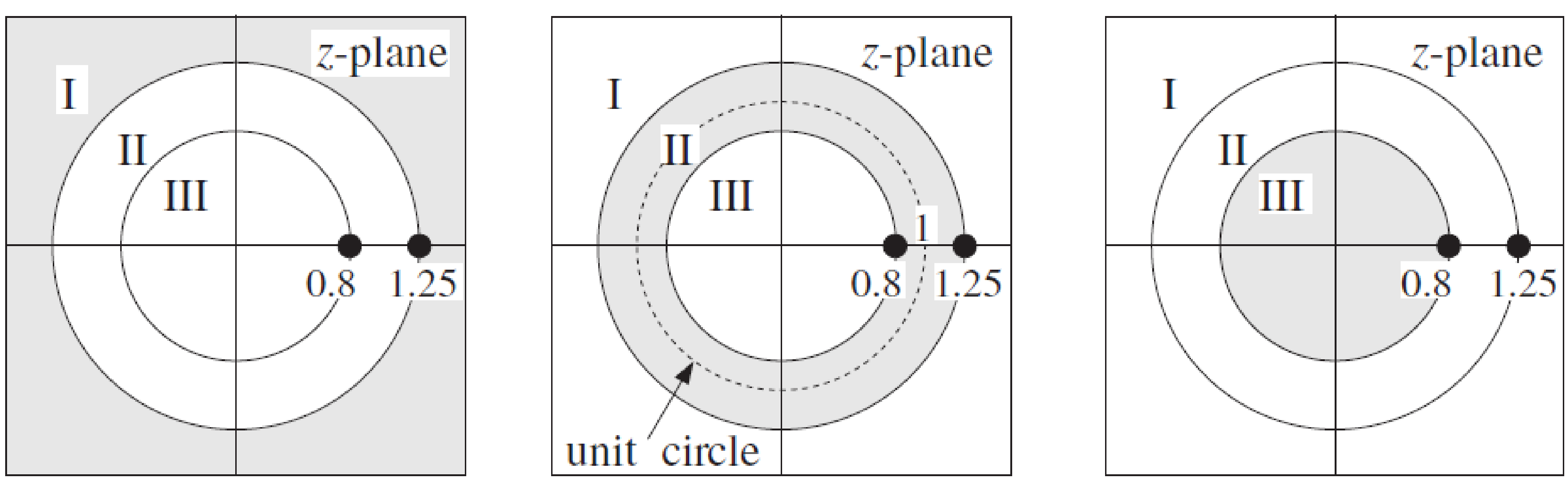
$f(n-D)$	$z^{-D}F(z)$	delay
$a^n f(n)$	$F(z/a)$	modulation
$nf(n)$	$-z\dfrac{dF(z)}{dz}$	z-differentiation
$f(n)*g(n)$	$F(z)G(z)$	convolution

Example 5.2.3: Determine the z-transform and corresponding region of convergence of the following signals:

1. $x(n)= (0.8)^n u(n)+(1.25)^n u(n)$
2. $x(n)= (0.8)^n u(n)-(1.25)^n u(-n-1)$
3. $x(n)= -(0.8)^n u(-n-1)-(1.25)^n u(-n-1)$
4. $x(n)= -(0.8)^n u(-n-1)+(1.25)^n u(n)$

Solution: Using Eq. (5.2.3) with $a = 0.8$ and $a = 1.25$, we note that the first three cases have exactly the *same* z-transform, namely,

$$X(z)=\frac{1}{1-0.8z^{-1}}+\frac{1}{1-1.25z^{-1}}=\frac{2-2.05z^{-1}}{1-2.05z^{-1}+z^{-2}}$$



causal and anti-causal ROC

$$x(n) = A_1 p_1^n u(n) + A_2 p_2^n u(n) + \cdots$$

$$|z| > \max_i |p_i|$$

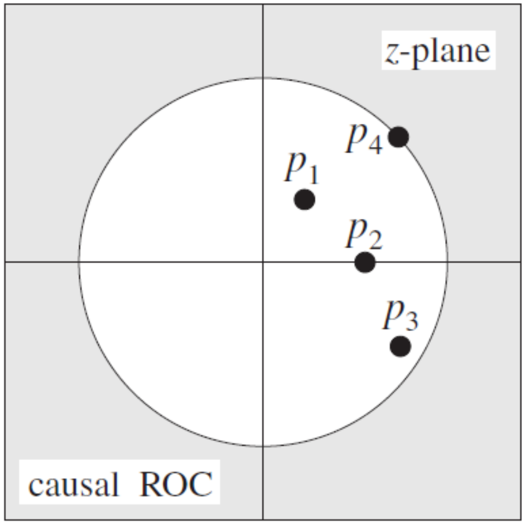
causal

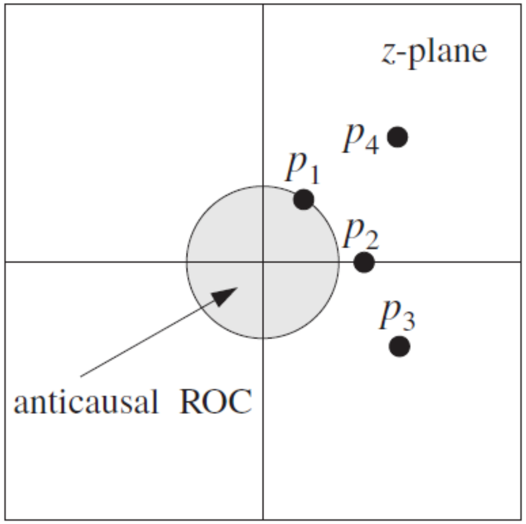
$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots$$

$$x(n) = -A_1 p_1^n u(-n-1) - A_2 p_2^n u(-n-1) - \cdots$$

$$|z| < \min_i |p_i|$$

anti-causal

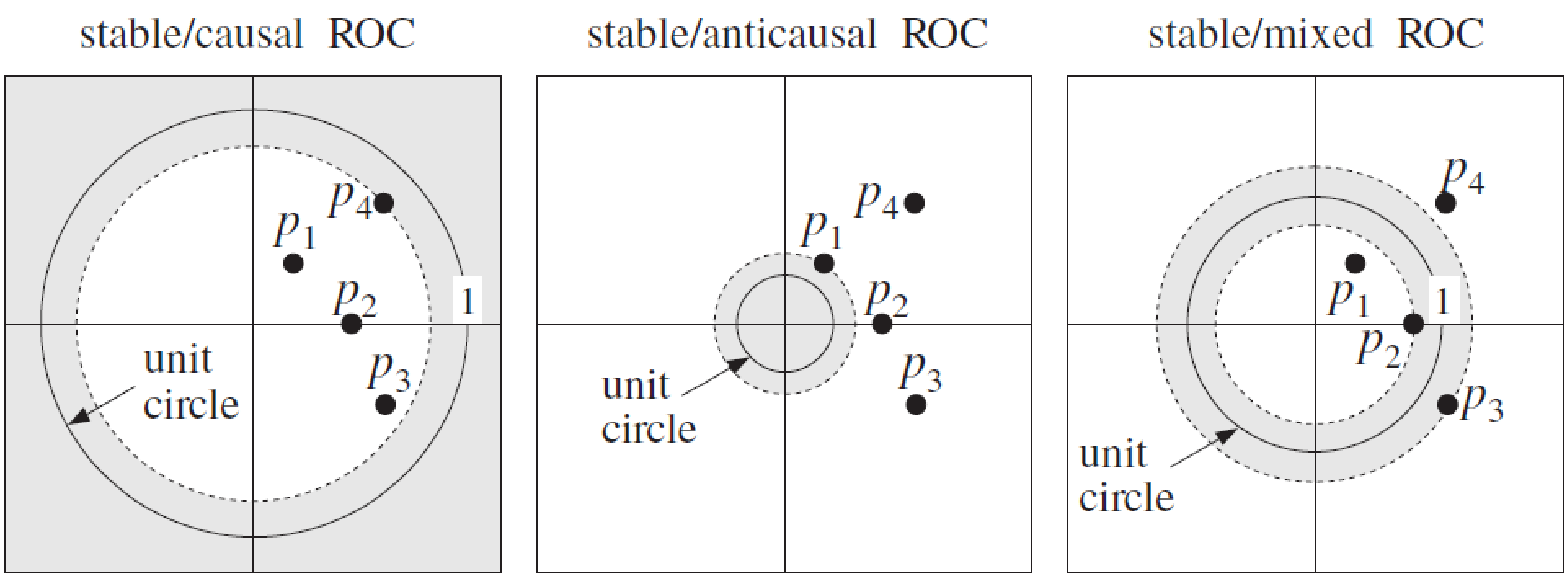




stable ROC

Theorem: A necessary and sufficient condition for the stability of a signal $x(n)$ is that is the ROC of its z-transform $X(z)$ contain the unit circle.

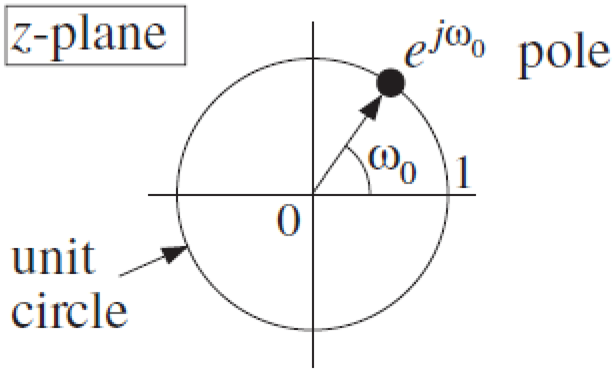
$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \frac{A_3}{1 - p_3 z^{-1}} + \frac{A_4}{1 - p_4 z^{-1}}$$



marginally-stable ROC

(causal)	$x(n) = e^{j\omega_0 n} u(n)$	$\lvert z \rvert > 1$
(anticausal)	$x(n) = -e^{j\omega_0 n} u(-n-1)$	$\lvert z \rvert < 1$

$$X(z) = \frac{1}{1 - e^{j\omega_0} z^{-1}}$$

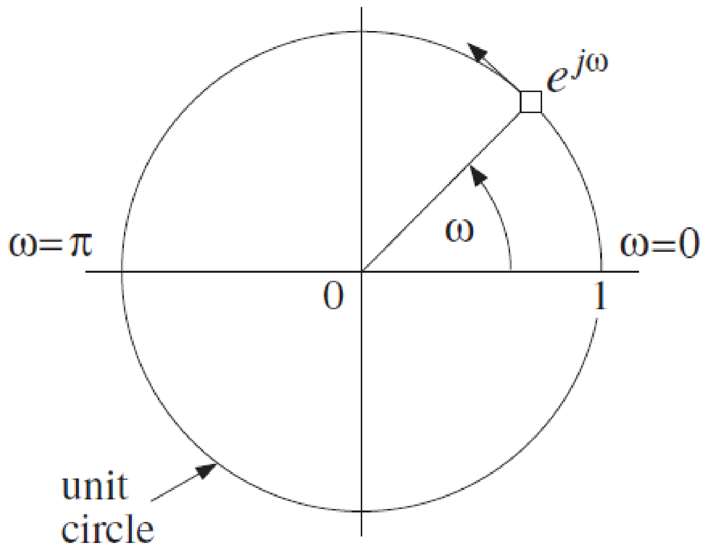


Frequency Spectrum

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \mathrm{DTFT}$$

strictly-speaking, it is defined only for stable signals

$$X(\omega) = X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}\Big|_{z=e^{j\omega}}$$

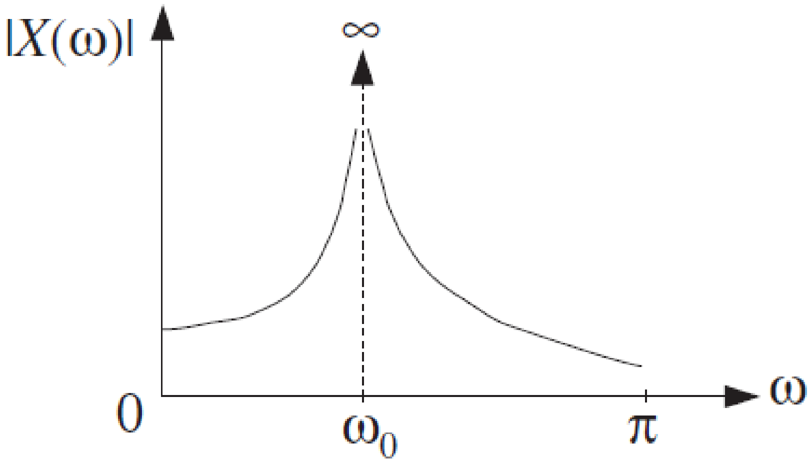


the frequency spectrum is the evaluation of the z-transform on the unit-circle

Frequency Spectrum

$$x(n) = e^{j\omega_0 n} u(n) \xrightarrow{z} X(z) = \frac{1}{1 - e^{j\omega_0} z^{-1}}$$

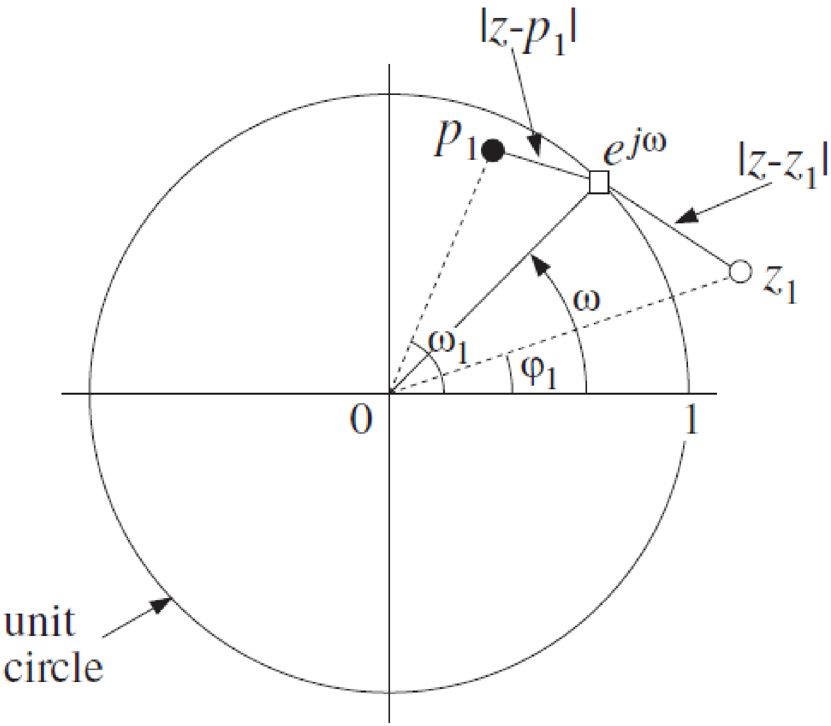
$$X(\omega) = \frac{1}{1 - e^{j\omega_0} e^{-j\omega}} = \frac{1}{1 - e^{j(\omega_0 - \omega)}}$$

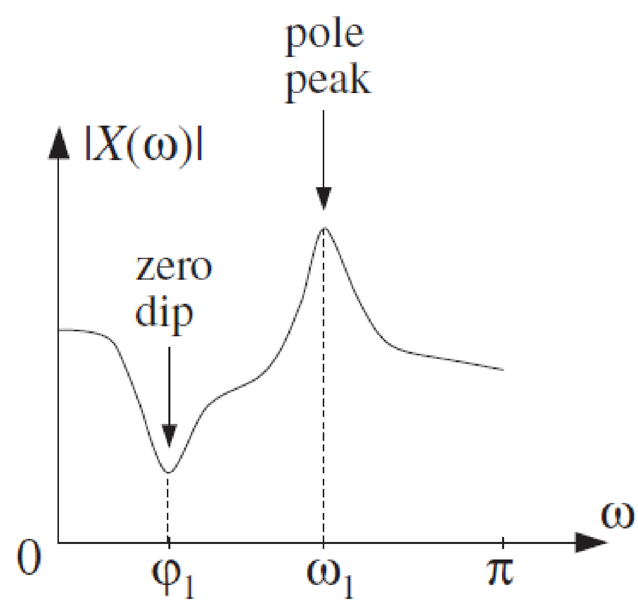


Pole-Zero Pattern

$$X(z) = \frac{1 - z_1 z^{-1}}{1 - p_1 z^{-1}} = \frac{z - z_1}{z - p_1}$$

$$X(\omega) = \frac{e^{j\omega} - z_1}{e^{j\omega} - p_1} \quad \Rightarrow \quad |X(\omega)| = \frac{|e^{j\omega} - z_1|}{|e^{j\omega} - p_1|}$$





partial fraction expansions (PFE)

$$X(z) = \frac{N(z)}{D(z)} = \frac{N(z)}{(1-p_1 z^{-1})(1-p_2 z^{-1})\cdots(1-p_M z^{-1})}$$

$$= \frac{A_1}{1-p_1 z^{-1}} + \frac{A_2}{1-p_2 z^{-1}} + \cdots + \frac{A_M}{1-p_M z^{-1}}$$

distinct poles

$$A_i = \left[(1-p_i z^{-1})X(z)\right]_{z=p_i} = \left[\frac{N(z)}{\prod_{j\neq i}(1-p_j z^{-1})}\right]_{z=p_i}$$

Example 5.5.2: In Example 5.2.3 the z-transform was written in the form

$$X(z) = \frac{2 - 2.05z^{-1}}{1 - 2.05z^{-1} + z^{-2}} = \frac{2 - 2.05z^{-1}}{(1 - 0.8z^{-1})(1 - 1.25z^{-1})}$$

Because the numerator polynomial has degree one in the variable z^{-1}, there is a PF expansion of the form:

$$X(z) = \frac{2 - 2.05z^{-1}}{(1 - 0.8z^{-1})(1 - 1.25z^{-1})} = \frac{A_1}{1 - 0.8z^{-1}} + \frac{A_2}{1 - 1.25z^{-1}}$$

The two coefficients are obtained by Eq. (5.5.2) as follows:

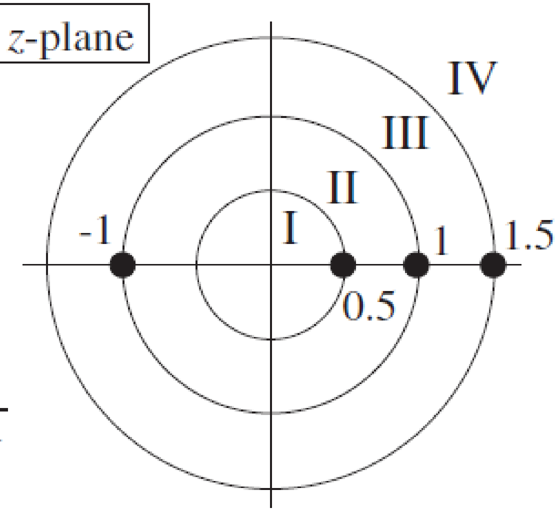
$$A_1 = \left[(1 - 0.8z^{-1})X(z)\right]_{z=0.8} = \left[\frac{2 - 2.05z^{-1}}{1 - 1.25z^{-1}}\right]_{z=0.8} = \frac{2 - 2.05/0.8}{1 - 1.25/0.8} = 1$$

$$A_2 = \left[(1 - 1.25z^{-1})X(z)\right]_{z=1.25} = \left[\frac{2 - 2.05z^{-1}}{1 - 0.8z^{-1}}\right]_{z=1.25} = 1$$

partial fraction expansions (PFE)

inverse z-transforms

Example 5.5.7: Determine all possible inverse z-transforms of



$$X(z) = \frac{7 - 9.5z^{-1} - 3.5z^{-2} + 5.5z^{-3}}{(1 - z^{-2})(1 - 0.5z^{-1})(1 - 1.5z^{-1})}$$

$$= \frac{1}{1 - z^{-1}} + \frac{1}{1 + z^{-1}} + \frac{3}{1 - 0.5z^{-1}} + \frac{2}{1 - 1.5z^{-1}}$$

$$x_1(n) = -[1 + (-1)^n + 3(0.5)^n + 2(1.5)^n]u(-n-1)$$

$$x_2(n) = 3(0.5)^n u(n) - [1 + (-1)^n + 2(1.5)^n]u(-n-1)$$

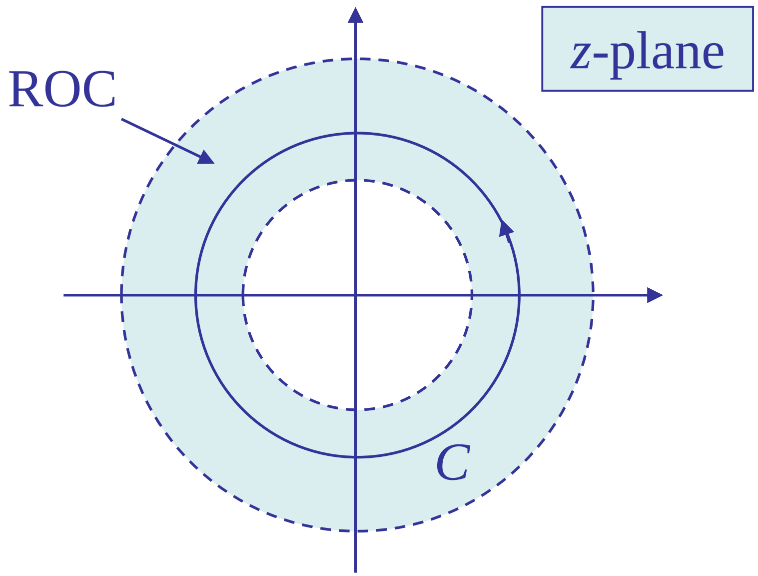
$$x_3(n) = [1 + (-1)^n + 3(0.5)^n]u(n) - 2(1.5)^n u(-n-1)$$

$$x_4(n) = [1 + (-1)^n + 3(0.5)^n + 2(1.5)^n]u(n)$$

inverse z-transforms by contour integration

$$x(n) = \oint_C X(z)\, z^n \frac{dz}{2\pi j z}$$

equals the sum of the residues of the poles of the integrand enclosed by the contour C



for stable signals,
C must be the unit-circle

inverse z-transforms by contour integration

$$\boxed{x(n) = \oint_C X(z)\, z^n \, \frac{dz}{2\pi j z}}$$

$$x(n) = \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, \frac{d\omega}{2\pi}$$

$$z = e^{j\omega} \quad \Rightarrow \quad \frac{dz}{2\pi j z} = \frac{d\omega}{2\pi}$$

for stable signals,
C must be the unit-circle

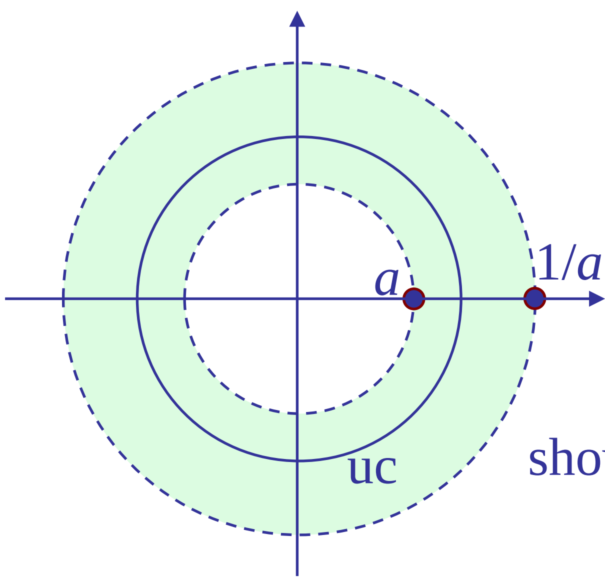
Example

inverse z-transforms by contour integration

$$-1 < a < 1$$

$$X(z) = \frac{1-a^2}{(1-az^{-1})(1-az)}\,, \quad \text{ROC } |a| < |z| < |a|^{-1}$$

$$x(n) = a^{|n|}\,, \qquad -\infty < n < \infty$$



showing the case $0 < a < 1$

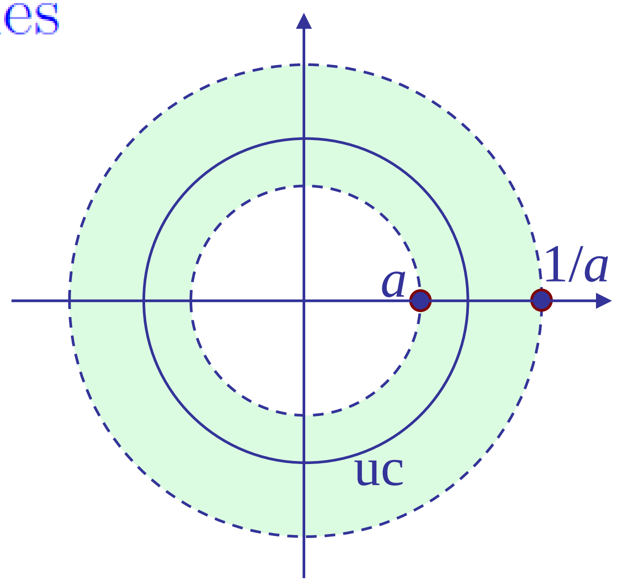
Example

inverse z-transforms by contour integration

$$X(z) = \frac{1-a^2}{(1-az^{-1})(1-az)} = \frac{(a-a^{-1})z}{(z-a)(z-a^{-1})}$$

$$x(n) = \oint_C X(z)\, z^n \frac{dz}{2\pi j z} = \oint_C \frac{(a-a^{-1})z^n}{(z-a)(z-a^{-1})} \frac{dz}{2\pi j}$$

$$= \text{sum of residues of enclosed poles}$$



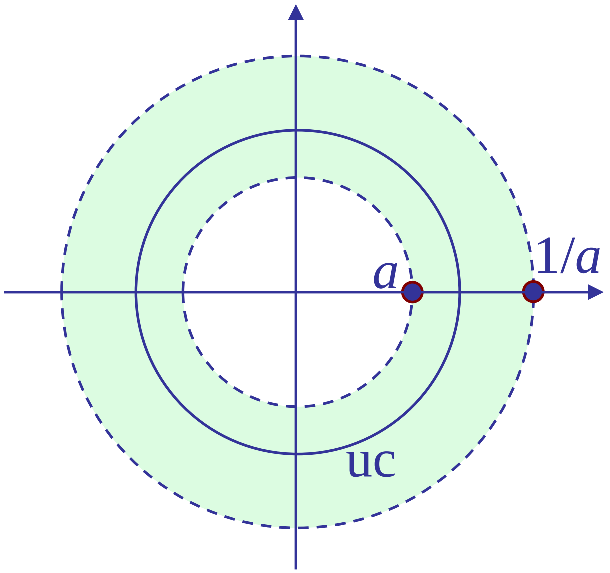
Example

inverse z-transforms by contour integration

$$x(n) = \oint_{\text{u.c.}} \frac{(a - a^{-1})z^n}{(z-a)(z-a^{-1})} \frac{dz}{2\pi j}$$

$$x(n) = \begin{cases} \text{Res}\{z=a\}\,, & \text{if } n \geq 0 \\ \text{Res}\{z=a\} + \text{Res}\{z=0\}\,, & \text{if } n < 0 \end{cases}$$

inconvenient because it is a multiple-pole residue



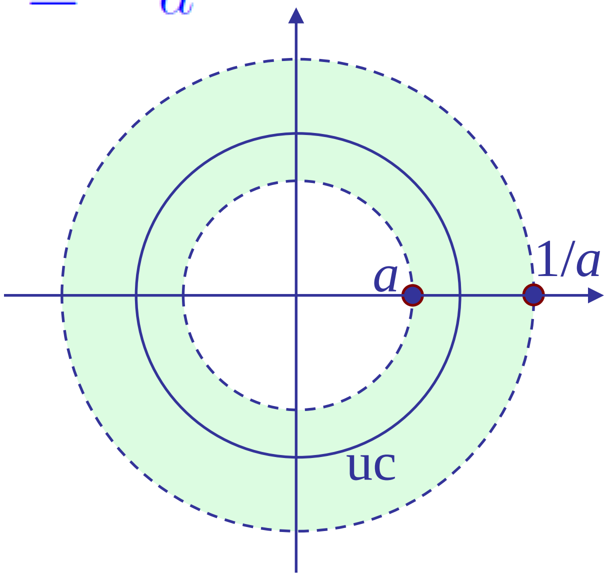
Example

inverse z-transforms by contour integration

$$x(n) = \oint_{\text{u.c.}} \frac{(a - a^{-1})z^n}{(z-a)(z-a^{-1})} \frac{dz}{2\pi j}$$

$$\text{Res}\{z = a\} = \left[\frac{(a - a^{-1})z^n}{(z-a^{-1})}\right]_{z=a} = a^n$$

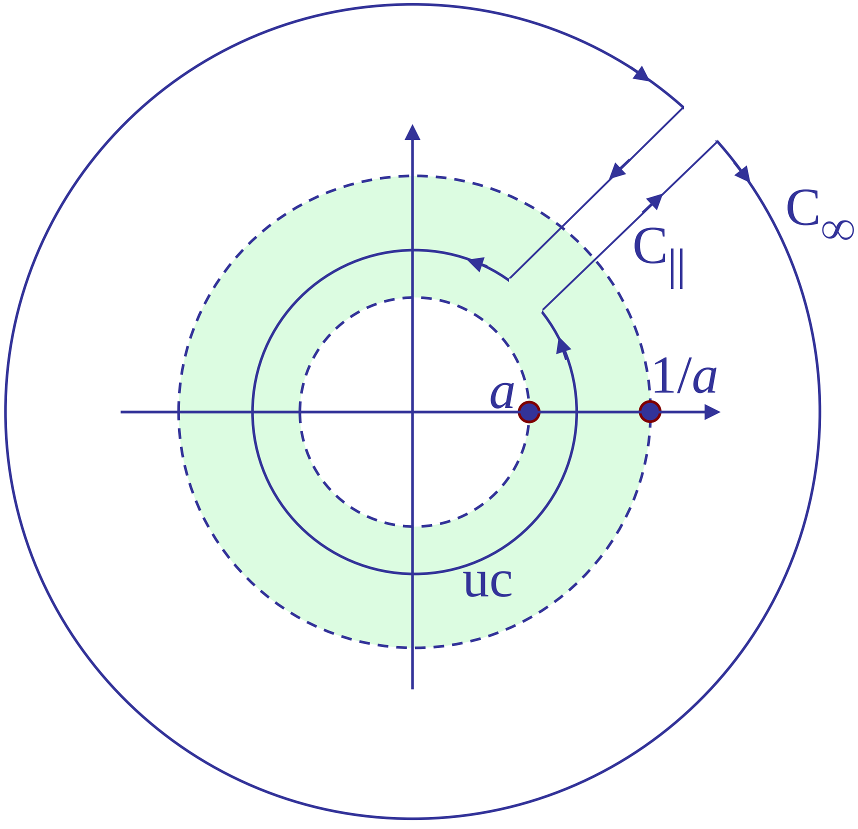
$$\text{Res}\{z = a^{-1}\} = \left[\frac{(a - a^{-1})z^n}{(z-a)}\right]_{z=a^{-1}} = -a^{-n}$$



Example

inverse z-transforms by contour integration

$$\oint_{\text{u.c.}} = \oint_{\text{u.c.}} + \oint_{C_{||}} + \oint_{C_\infty} = \oint_{C_{\text{extended}}}$$



Example

inverse z-transforms by contour integration

$$z = Re^{j\theta}, \quad dz = jRe^{j\theta}d\theta$$

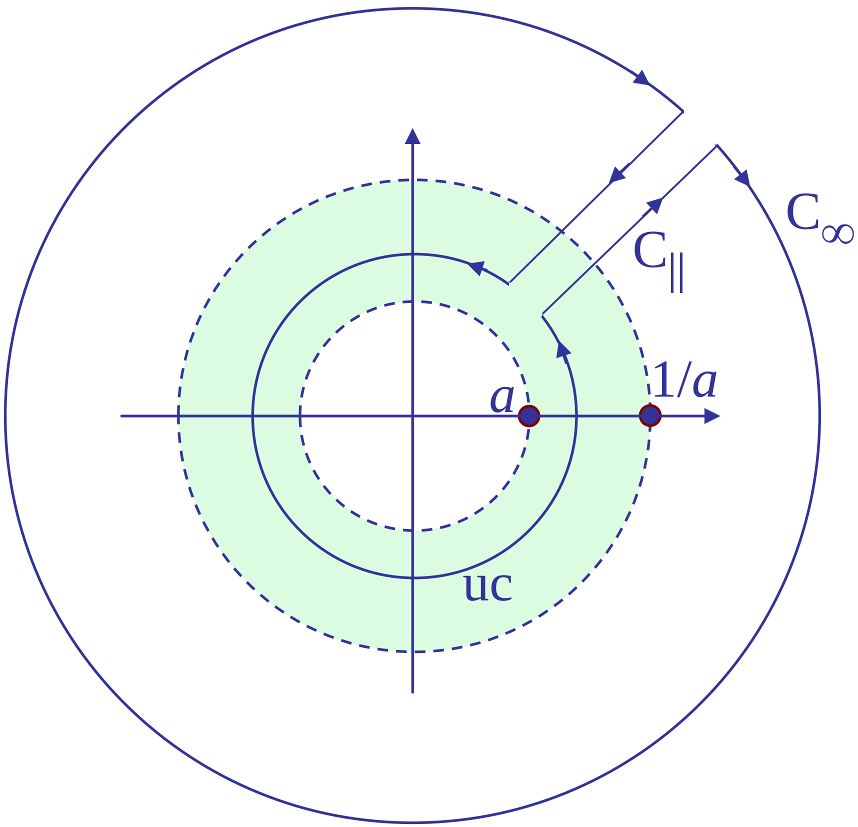
$$\oint_{C_\infty} = \text{const} \cdot \lim_{R\to\infty}\left[\frac{(a-a^{-1})z^n}{(z-a)(z-a^{-1})}\right]$$

$$= \text{const} \cdot \lim_{R\to\infty}\left[R^{n-1}\right] = 0$$

$$\text{if } n < 0$$

$$x(n) = -\text{Res}\{z = a^{-1}\} = a^{-n}$$

because C_{ext} is now clock-wise



Example

inverse z-transforms by contour integration

$$x(n) = a^{|n|} = \begin{cases} a^n, & n \geq 0 \\ a^{-n}, & n < 0 \end{cases}$$

$$X(z) = \frac{1-a^2}{(1-az^{-1})(1-az)} = \frac{(a-a^{-1})z^{-1}}{(1-az^{-1})(1-a^{-1}z^{-1})}$$

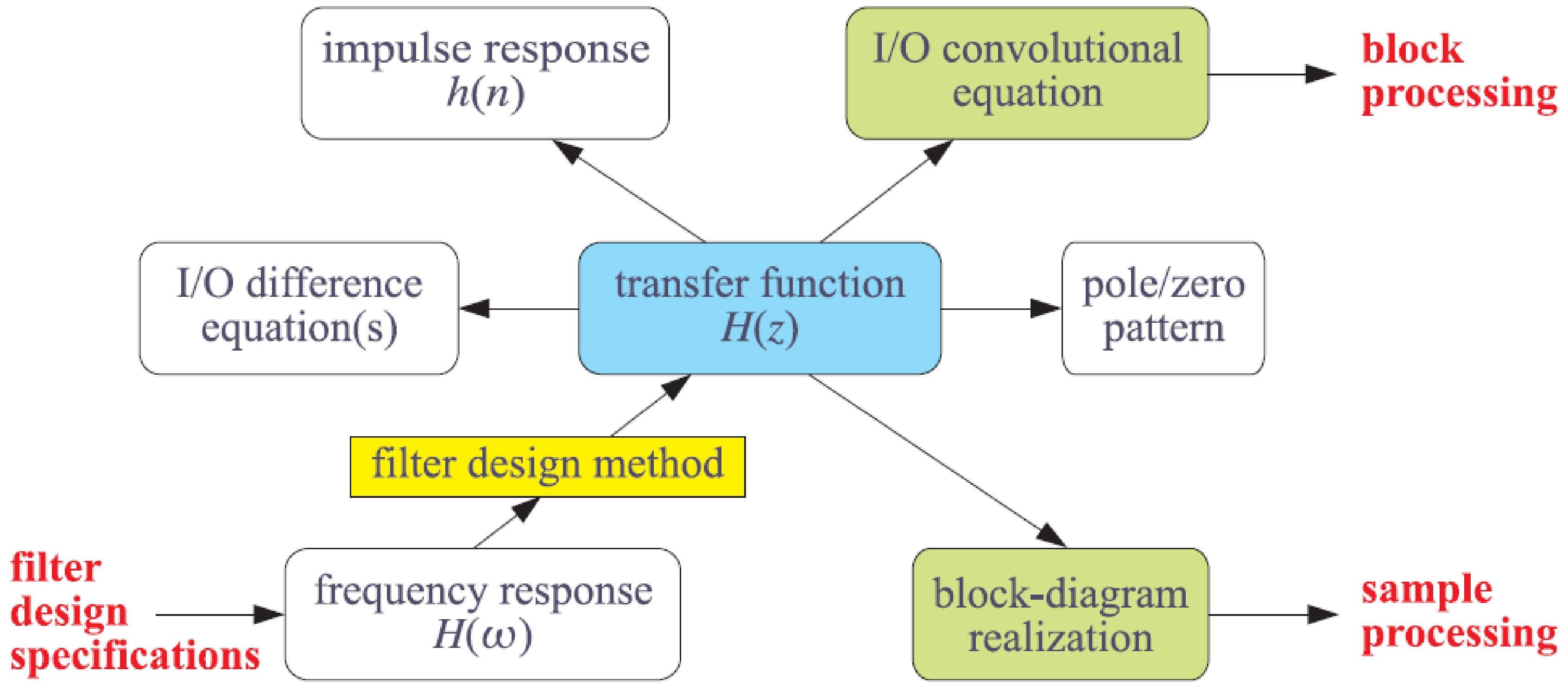
$$= \frac{1}{1-az^{-1}} - \frac{1}{1-a^{-1}z^{-1}}$$

PFE

$$\text{ROC} \quad |a| < |z| < |a^{-1}|$$

$$\boxed{x(n) = a^n u(n) + a^{-n} u(-n-1) = a^{|n|}}$$

Transfer Functions



Example

$$H(z) = \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}}$$

transfer function

$$H(z) = \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}} = A_0 + \frac{A_1}{1 - 0.8z^{-1}} = -2.5 + \frac{7.5}{1 - 0.8z^{-1}}$$

$$A_0 = H(z)\big|_{z=0} = \left.\frac{5 + 2z^{-1}}{1 - 0.8z^{-1}}\right|_{z=0} = \left.\frac{5z + 2}{z - 0.8}\right|_{z=0} = \frac{2}{-0.8} = -2.5$$

$$A_1 = (1 - 0.8z^{-1})H(z)\big|_{z=0.8} = (5 + 2z^{-1})\big|_{z=0.8} = 5 + 2/0.8 = 7.5$$

$$h(n) = -2.5\delta(n) + 7.5(0.8)^n u(n)$$

impulse response

Example

$$H(z) = \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}}$$

transfer function

$$(1 - 0.8z^{-1})H(z) = 5 + 2z^{-1} \quad \Rightarrow \quad H(z) = 0.8z^{-1}H(z) + 5 + 2z^{-1}$$

$$h(n) = 0.8h(n-1) + 5\delta(n) + 2\delta(n-1)$$

difference equation for *h*(*n*)

$$\begin{aligned} y_n &= h_0 x_n + h_1 x_{n-1} + h_2 x_{n-2} + h_3 x_{n-3} + \cdots \\ &= 5x_n + 7.5\big[(0.8)x_{n-1} + (0.8)^2 x_{n-2} + (0.8)^3 x_{n-3} + \cdots\big] \end{aligned}$$

convolutional input/output equation

Example

$$H(z) = \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}}$$

transfer function

$$Y(z) = H(z)X(z) = \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}}X(z) \quad \Rightarrow \quad (1 - 0.8z^{-1})Y(z) = (5 + 2z^{-1})X(z)$$

$$Y(z) - 0.8z^{-1}Y(z) = 5X(z) + 2z^{-1}X(z)$$

$$y(n) - 0.8y(n-1) = 5x(n) + 2x(n-1)$$

$$y(n) = 0.8y(n-1) + 5x(n) + 2x(n-1)$$

input/output difference equation

Example

$$H(z) = \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}}$$

transfer function

frequency response

$$H(z) = \frac{5(1 + 0.4z^{-1})}{1 - 0.8z^{-1}} \Rightarrow H(\omega) = \frac{5(1 + 0.4e^{-j\omega})}{1 - 0.8e^{-j\omega}}$$

magnitude response

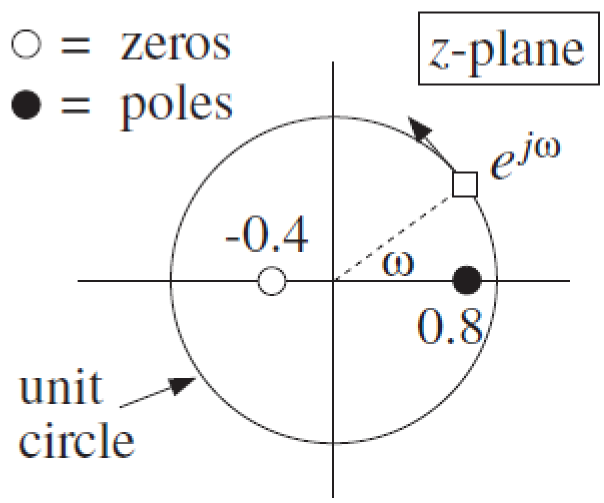
$$|H(\omega)| = \frac{5\sqrt{1 + 0.8\cos\omega + 0.16}}{\sqrt{1 - 1.6\cos\omega + 0.64}}$$

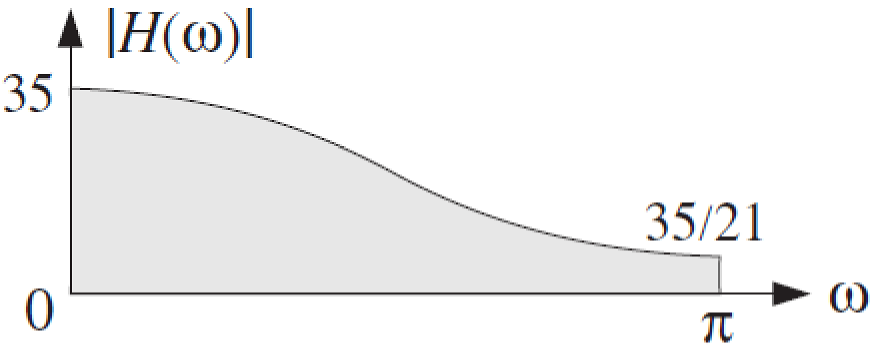
Example

$$H(z) = \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}}$$

transfer function

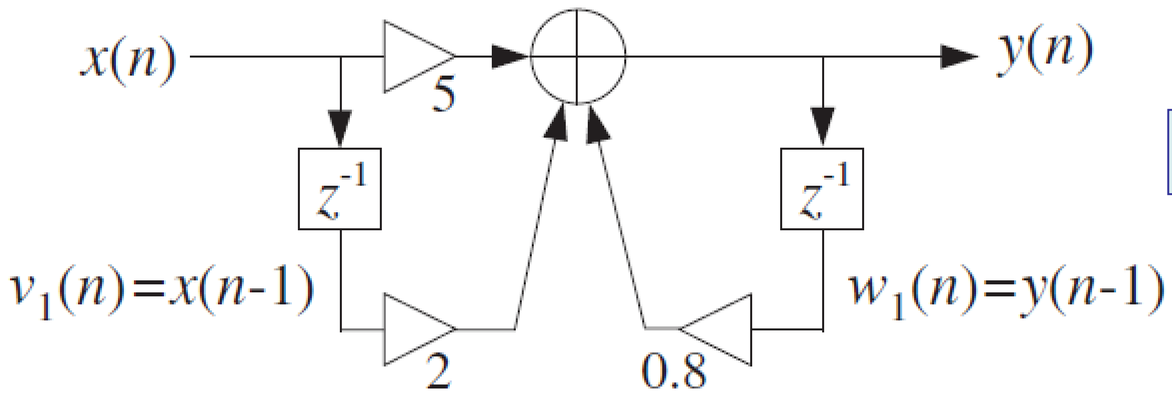
$$|H(\omega)| = \frac{5\sqrt{1 + 0.8\cos\omega + 0.16}}{\sqrt{1 - 1.6\cos\omega + 0.64}}$$





Example

$$y(n) = 0.8y(n-1) + 5x(n) + 2x(n-1)$$



direct-form realization

$$v_1(n+1) = x(n)$$

$$w_1(n+1) = y(n)$$

state updating

must initialize v_1, w_1

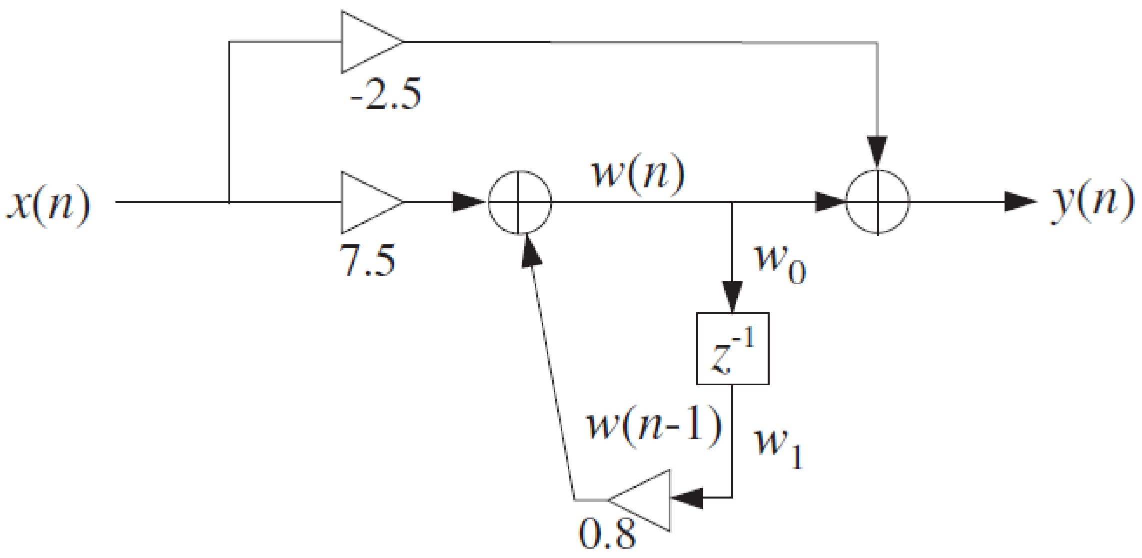
for each input sample x do:

$$y = 0.8w_1 + 5x + 2v_1$$
$$v_1 = x$$
$$w_1 = y$$

sample processing algorithm

Example

$$H(z) = \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}} = -2.5 + \frac{7.5}{1 - 0.8z^{-1}}$$



parallel realization

$$w_1(n+1) = w_0(n)$$

state updating

must initialize w_1

$$\begin{aligned}
&\textit{for each input sample } x \textit{ do:}\\
&\quad w_0 = 0.8w_1 + 7.5x\\
&\quad y = w_0 - 2.5x\\
&\quad w_1 = w_0
\end{aligned}$$

sample processing algorithm

Example

$$Y(z) = H(z)X(z) = \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}} X(z)$$

$$W(z) = \frac{1}{1 - 0.8z^{-1}} X(z)$$

$$Y(z) = (5 + 2z^{-1})W(z)$$

$$\Rightarrow$$

$$w(n) = 0.8w(n-1) + x(n)$$

$$y(n) = 5w(n) + 2w(n-1)$$

$$w_0(n) = w(n)$$

$$w_1(n) = w(n-1)$$

$$\Rightarrow \quad w_1(n+1) = w_0(n)$$

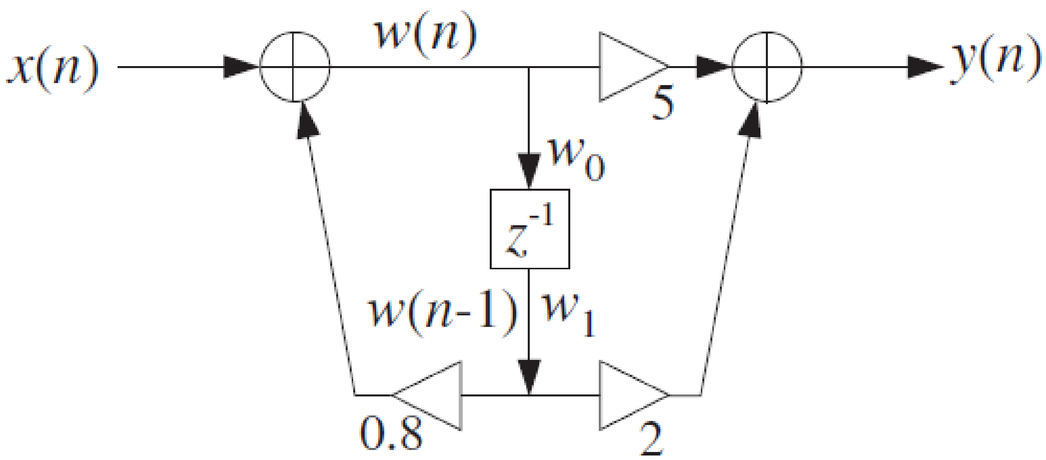
$$w_0(n) = 0.8w_1(n) + x(n)$$

$$y(n) = 5w_0(n) + 2w_1(n)$$

$$w_1(n+1) = w_0(n)$$

canonical realization
direct-form II

Example



canonical realization direct-form II

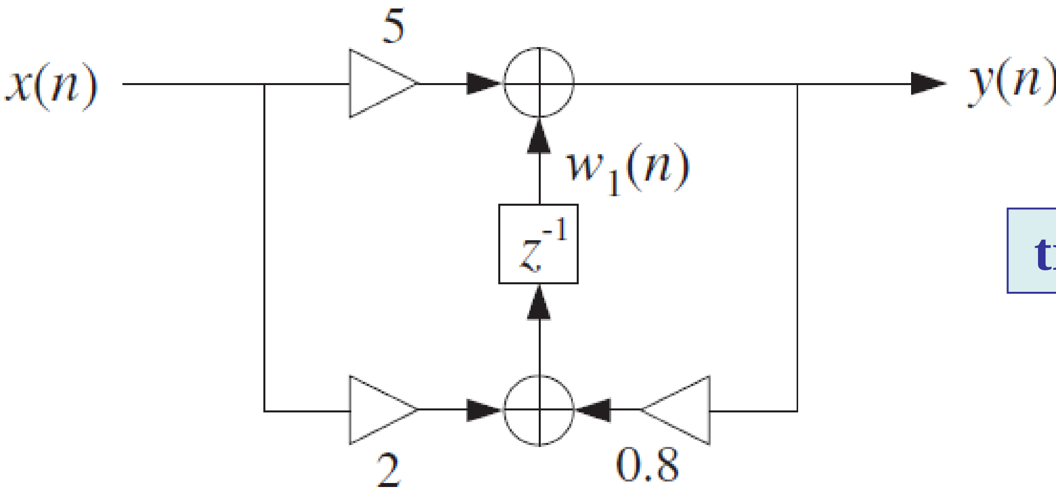
must initialize w_1

for each input sample x *do:*

$$w_0 = 0.8w_1 + x$$
$$y = 5w_0 + 2w_1$$
$$w_1 = w_0$$

sample processing algorithm

Example



transposed realization

$$w_1(n) = 2x(n-1) + 0.8y(n-1)$$
$$w_1(n+1) = 2x(n) + 0.8y(n)$$

state updating

must initialize w_1

for each input sample x do:
$$y = w_1 + 5x$$
$$w_1 = 2x + 0.8y$$

sample processing algorithm

Sinusoidal Response

$$x(n) = e^{j\omega_0 n}, \quad -\infty < n < \infty$$

$$y(n) = \sum_m h(m)x(n-m) = \sum_m h(m)e^{j(n-m)\omega_0} = e^{j\omega_0 n}\sum_m h(m)e^{-j\omega_0 m}$$

$$y(n) = H(\omega_0)e^{j\omega_0 n}$$

$$\boxed{e^{j\omega_0 n} \xrightarrow{H} H(\omega_0)e^{j\omega_0 n}}$$

steady-state sinusoidal response

$$\cos(\omega_0 n) \xrightarrow{H} |H(\omega_0)|\cos\left(\omega_0 n + \arg H(\omega_0)\right)$$

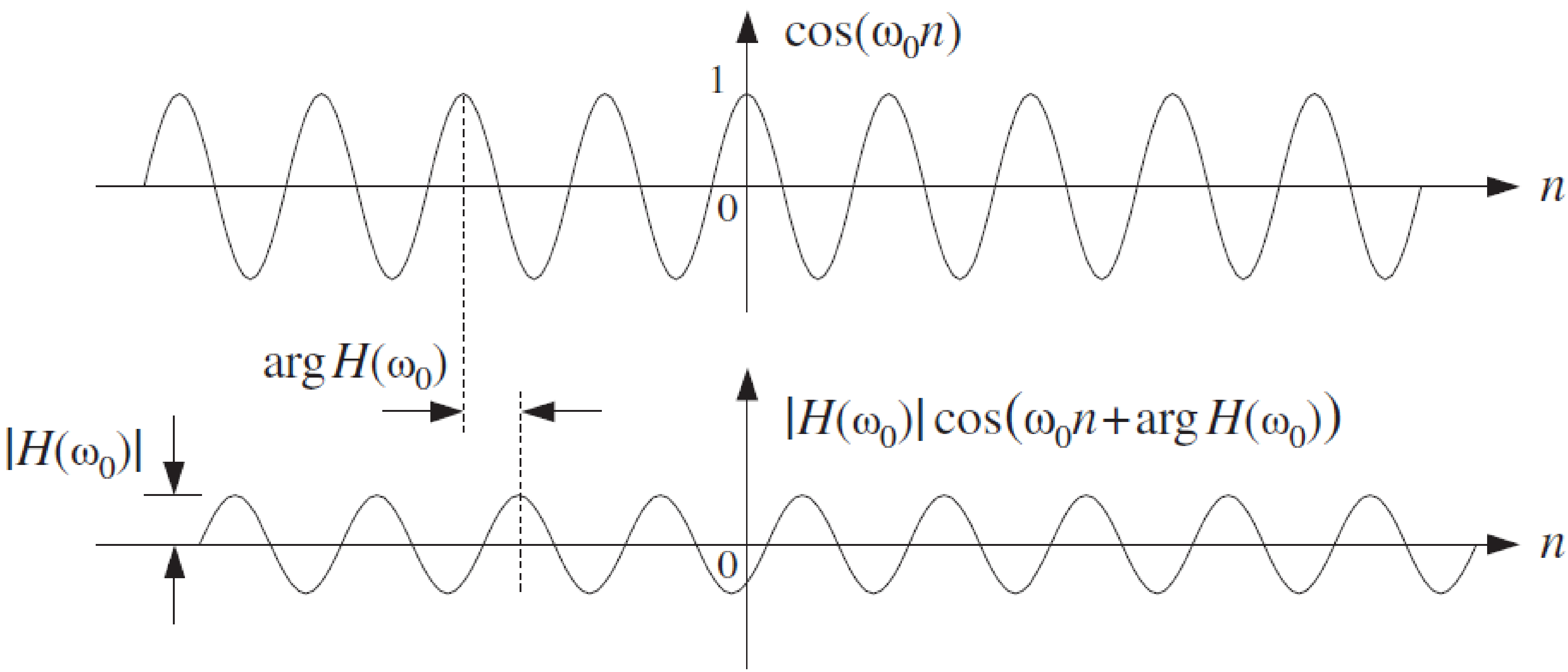
$$\sin(\omega_0 n) \xrightarrow{H} |H(\omega_0)|\sin\left(\omega_0 n + \arg H(\omega_0)\right)$$

Sinusoidal Response

$$\cos(\omega_0 n) \xrightarrow{H} |H(\omega_0)| \cos\big(\omega_0 n + \arg H(\omega_0)\big)$$

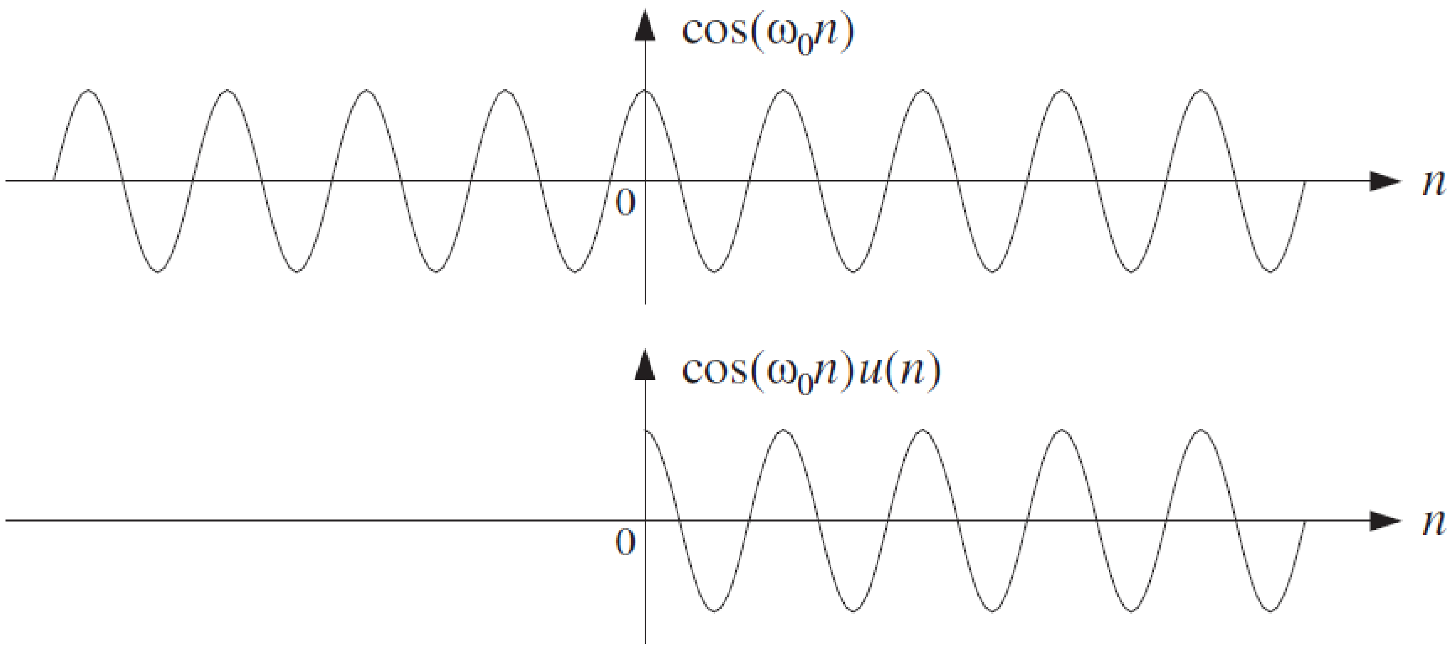
$$\sin(\omega_0 n) \xrightarrow{H} |H(\omega_0)| \sin\big(\omega_0 n + \arg H(\omega_0)\big)$$

steady-state



Sinusoidal Response

transient response



double-sided vs. one-sided sinusoids

transient response

$$x(n) = e^{j\omega_0 n}u(n) \xrightarrow{z} X(z) = \frac{1}{1 - e^{j\omega_0}z^{-1}}$$

$$\boxed{\text{ROC } |z| > |e^{j\omega_0}| = 1}$$

$$H(z) = \frac{N(z)}{D(z)} = \frac{N(z)}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})\cdots(1 - p_M z^{-1})}$$

$$\boxed{|p_i| < 1}$$

stable and causal *H*(*z*) with distinct poles strictly inside the UC, and deg*N* < *M*+1

$$Y(z) = H(z)X(z) = \frac{N(z)}{(1 - e^{j\omega_0}z^{-1})(1 - p_1 z^{-1})(1 - p_2 z^{-1})\cdots(1 - p_M z^{-1})}$$

$$Y(z) = \frac{C}{1 - e^{j\omega_0}z^{-1}} + \frac{B_1}{1 - p_1 z^{-1}} + \frac{B_2}{1 - p_2 z^{-1}} + \cdots + \frac{B_M}{1 - p_M z^{-1}}$$

$$\boxed{C = H(z)\big|_{z=e^{j\omega_0}} = H(\omega_0)}$$

transient response

$$C = (1 - e^{j\omega_0} z^{-1}) Y(z) \big|_{z=e^{j\omega_0}} = \left[(1 - e^{j\omega_0} z^{-1}) \frac{H(z)}{1 - e^{j\omega_0} z^{-1}} \right]_{z=e^{j\omega_0}}$$

$$\boxed{C = H(z) \big|_{z=e^{j\omega_0}} = H(\omega_0)}$$

$$Y(z) = \frac{H(\omega_0)}{1 - e^{j\omega_0} z^{-1}} + \frac{B_1}{1 - p_1 z^{-1}} + \frac{B_2}{1 - p_2 z^{-1}} + \cdots + \frac{B_M}{1 - p_M z^{-1}}$$

$$\boxed{y(n) = H(\omega_0) e^{j\omega_0 n} + B_1 p_1^n + B_2 p_2^n + \cdots + B_M p_M^n} \quad n \geq 0$$

$$y(n) \to H(\omega_0) e^{j\omega_0 n} \quad \text{as} \quad n \to \infty \qquad |p_i| < 1$$

steady-state response

transient response

$$\boxed{y(n) = H(\omega_0)\, e^{j\omega_0 n} + B_1 p_1^n + B_2 p_2^n + \cdots + B_M p_M^n} \quad n \geq 0; \qquad |p_i| < 1$$

$$\rho = \max_i |p_i|$$

pole of largest radius, or, closest to the UC from inside

$$\rho^{n_{\text{eff}}} = \epsilon$$

$$\boxed{n_{\text{eff}} = \frac{\ln \epsilon}{\ln \rho} = \frac{\ln(1/\epsilon)}{\ln(1/\rho)}}$$

time constant in samples

$$\epsilon = 1\% = 0.01$$

$$\epsilon = 0.1\% = 10^{-3}$$

40-dB and 60-dB time constants

Example

For a causal and an anti-causal sinusoidal input, determine the corresponding output of the following causal/stable system

$$h(n) = a^n u(n)\,, \qquad 0 < a < 1$$

$$H(z) = \frac{1}{1 - az^{-1}}\,, \quad \text{ROC } |z| > a$$

$$x_1(n) = e^{j\omega_0 n} u(n) \longrightarrow \boxed{H(z)} \longrightarrow y_1(n) = \,?$$

$$x_2(n) = e^{j\omega_0 n} u(-n-1) \longrightarrow \boxed{H(z)} \longrightarrow y_2(n) = \,?$$

$$x(n) = x_1(n) + x_2(n) \longrightarrow \boxed{H(z)} \longrightarrow y(n) = \,?$$

Example

$$X_1(z) = \frac{1}{1 - e^{j\omega_0} z^{-1}}, \qquad \text{ROC } |z| > 1$$

$$X_2(z) = -\frac{1}{1 - e^{j\omega_0} z^{-1}}, \qquad \text{ROC } |z| < 1$$

$$Y_1(z) = \frac{1}{(1 - e^{j\omega_0} z^{-1})(1 - a z^{-1})}, \qquad \text{ROC } |z| > 1$$

$$Y_2(z) = -\frac{1}{(1 - e^{j\omega_0} z^{-1})(1 - a z^{-1})}, \qquad \text{ROC } a < |z| < 1$$

Example

$$Y_1(z) = \frac{1}{(1-e^{j\omega_0}z^{-1})(1-az^{-1})}, \qquad \text{ROC } |z|>1$$

$$Y_2(z) = -\frac{1}{(1-e^{j\omega_0}z^{-1})(1-az^{-1})}, \qquad \text{ROC } a<|z|<1$$

$$Y_1(z) = \frac{1}{(1-e^{j\omega_0}z^{-1})(1-az^{-1})} = \frac{H(\omega_0)}{1-e^{j\omega_0}z^{-1}} + \frac{B}{1-az^{-1}}$$

$$Y_2(z) = -\frac{1}{(1-e^{j\omega_0}z^{-1})(1-az^{-1})} = -\frac{H(\omega_0)}{1-e^{j\omega_0}z^{-1}} - \frac{B}{1-az^{-1}}$$

$$H(\omega_0) = \frac{1}{1-ae^{-j\omega_0}}, \qquad B = \frac{1}{1-e^{j\omega_0}a^{-1}}$$

Example

$$Y_1(z) = \frac{1}{(1-e^{j\omega_0}z^{-1})(1-az^{-1})} = \frac{H(\omega_0)}{1-e^{j\omega_0}z^{-1}} + \frac{B}{1-az^{-1}}$$

$$Y_2(z) = -\frac{1}{(1-e^{j\omega_0}z^{-1})(1-az^{-1})} = -\frac{H(\omega_0)}{1-e^{j\omega_0}z^{-1}} - \frac{B}{1-az^{-1}}$$

$$\text{ROC } |z| > 1$$

$$\text{ROC } a < |z| < 1$$

$$y_1(n) = H(\omega_0)e^{j\omega_0 n}u(n) + Ba^n u(n)$$

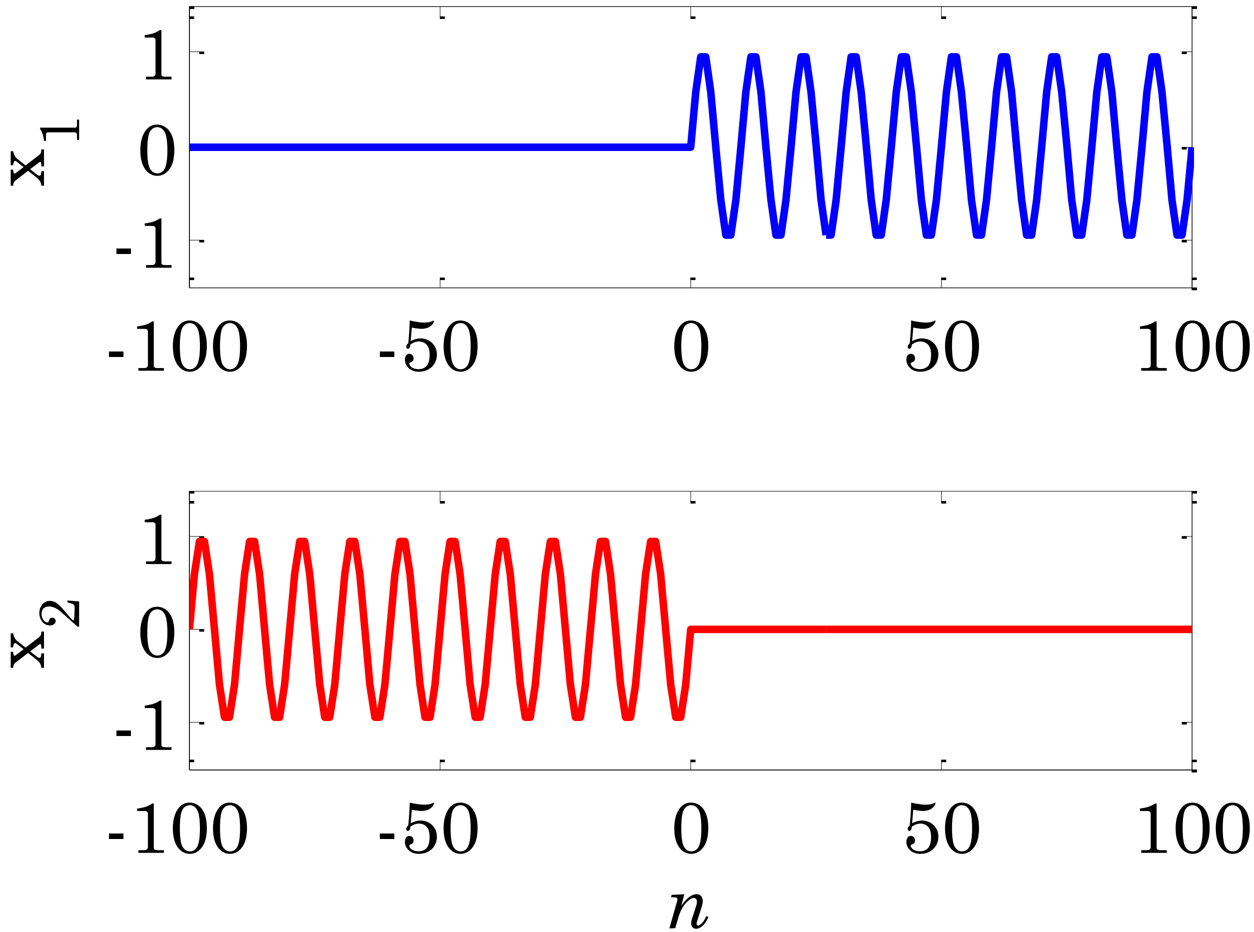
$$y_2(n) = H(\omega_0)e^{j\omega_0 n}u(-n-1) - Ba^n u(n)$$

$$x_1(n) + x_2(n) = e^{j\omega_0 n}$$

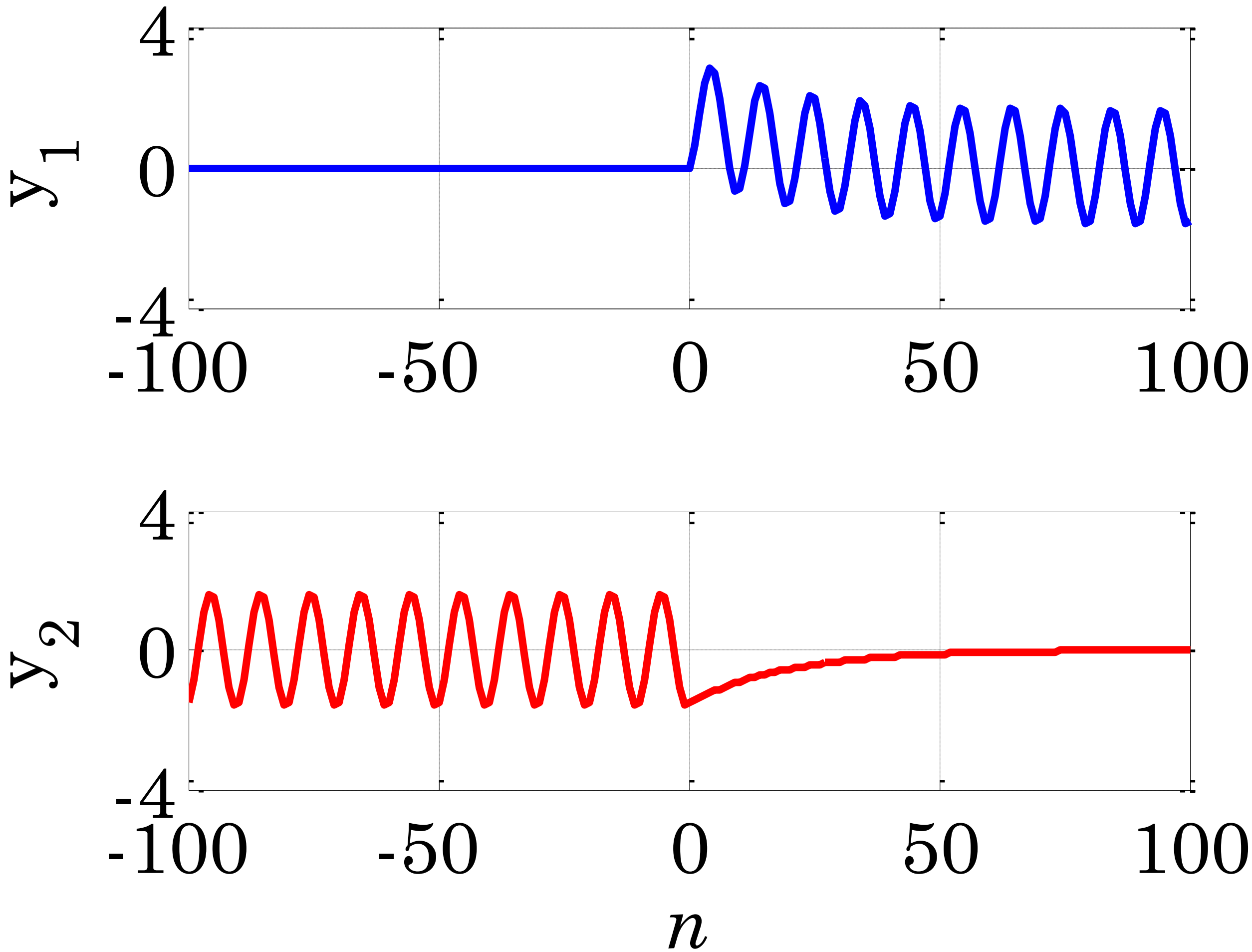
$$y_1(n) + y_2(n) = H(\omega_0)e^{j\omega_0 n}$$

Example

$\omega_0 = \pi/5, \quad a = 0.95$



Example



Pole-Zero Placement

see the files

notch-digital.pdf

notch-analog.pdf