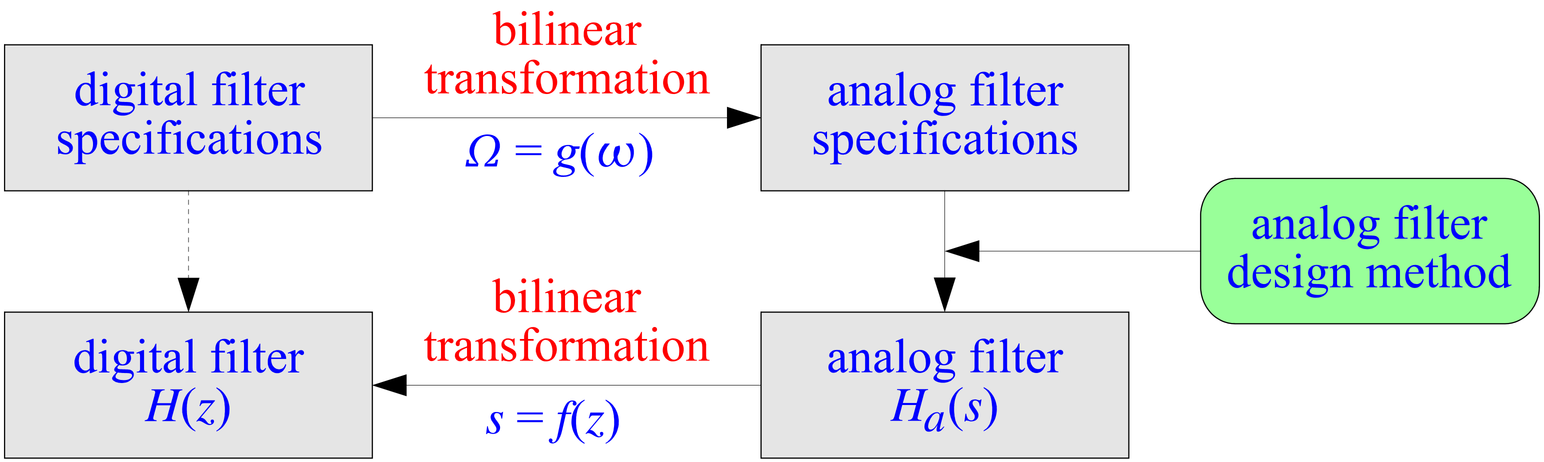
DSA – March 29, 2021

Topics: Digital filter design, IIR filter design, bilinear transformation method, notch, peaking, audio EQ filters, higher-order filters, Butterworth, Chebyshev, Elliptic, FIR filter design, window method, Kaiser windows, frequency-sampling method.



$$\boxed{H(z) = H_a(s)\Big|_{s=f(z)} = H_a\big(f(z)\big)}$$

$$\boxed{H(\omega) = H_a(\Omega)\Big|_{\Omega=g(\omega)} = H_a\big(g(\omega)\big)}$$

For a more unified and complete discussion that includes all cases, Butterworth, Chebyshev, and Elliptic, see the handout, **notes.pdf** with associated MATLAB functions included in, **notes-mfiles.zip**.

Elliptic filters, also known as, Cauer or Zolotarev filters, achieve the smallest filter order for the same specifications, or, the narrowest transition width for the same filter order, as compared to other filter types.

On the negative side, they have the most nonlinear phase response over their passband. The following table compares the basic filter types with regard to filter order and phase response.

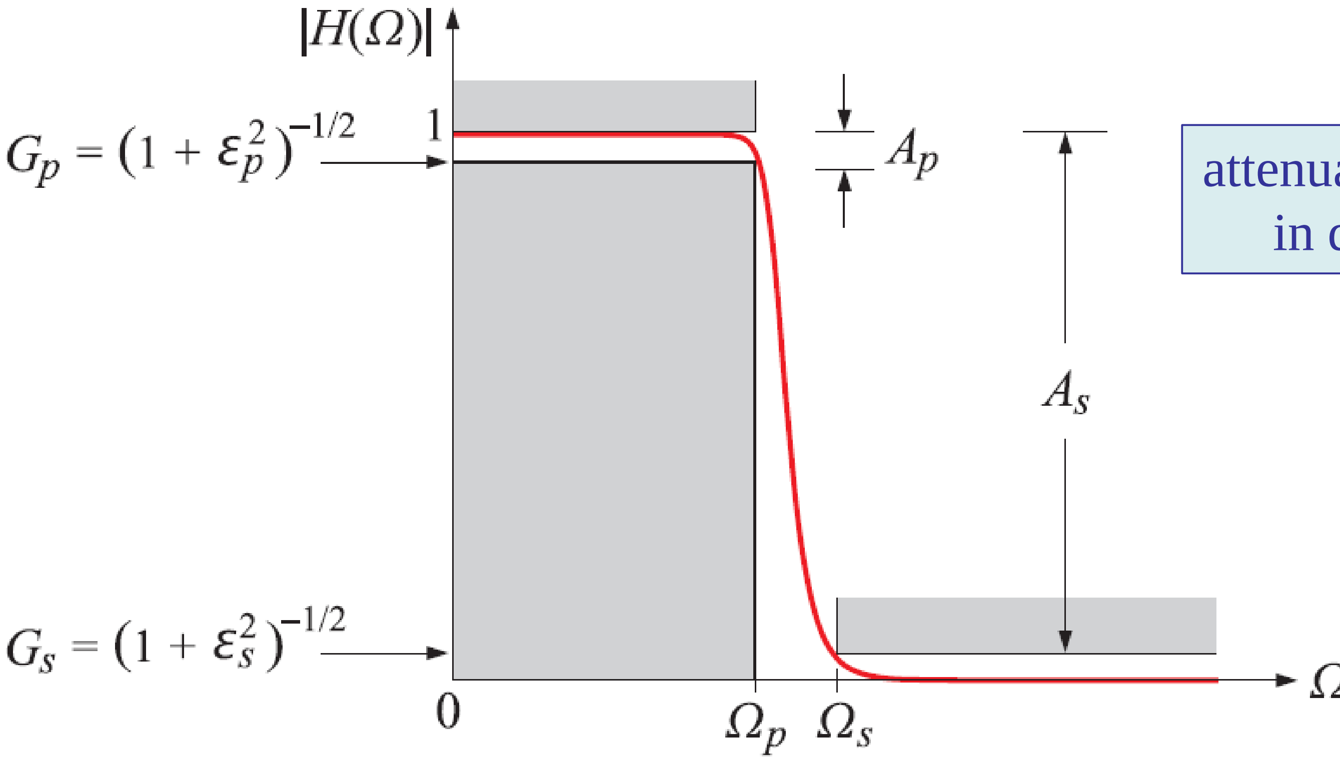
smaller order, or, narrower transition ↓		more linear phase over their passband ↑
	Bessel	
	Butterworth	
	Chebyshev	
	Elliptic	

Bessel filters have, by design, the most linear phase in their passband but the slowest rolloff – we will consider them in I2SP-Ch.12.

summary – analog designs

passband gain

stopband gain



attenuations in dB

Brick wall specifications for a Butterworth filter.

summary – analog designs

passband and stopband gains

$$G_p = \frac{1}{\sqrt{1+\varepsilon_p^2}} = 10^{-A_p/20}\,, \qquad G_s = \frac{1}{\sqrt{1+\varepsilon_s^2}} = 10^{-A_s/20}$$

attenuations in dB

ripple parameters

$$A_p = -20\log_{10} G_p = 10\log_{10}(1+\varepsilon_p^2)$$
$$A_s = -20\log_{10} G_s = 10\log_{10}(1+\varepsilon_s^2)$$

$$\Rightarrow$$

$$\varepsilon_p = \sqrt{10^{A_p/10}-1}$$
$$\varepsilon_s = \sqrt{10^{A_s/10}-1}$$

$$k = \frac{\Omega_p}{\Omega_s}\,, \qquad k_1 = \frac{\varepsilon_p}{\varepsilon_s}\,, \qquad \text{typically: } \begin{cases} k \lesssim 1 \\ k_1 \ll 1 \end{cases}$$

discrimination and selectivity parameters

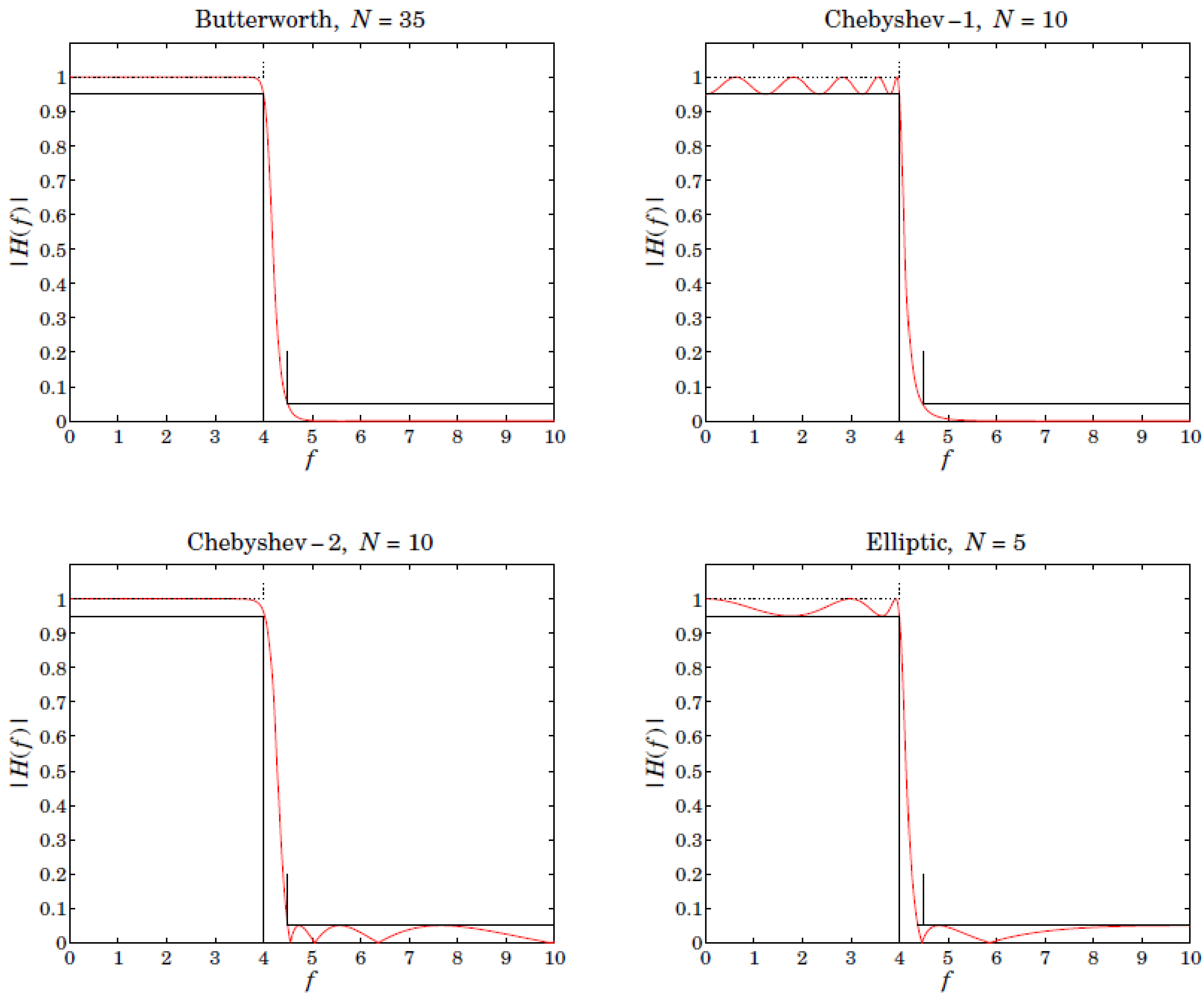
summary – analog designs

$$|H(\Omega)|^2 = \frac{1}{1 + \varepsilon_p^2 F_N^2(w)}\,, \quad w = \frac{\Omega}{\Omega_p}$$

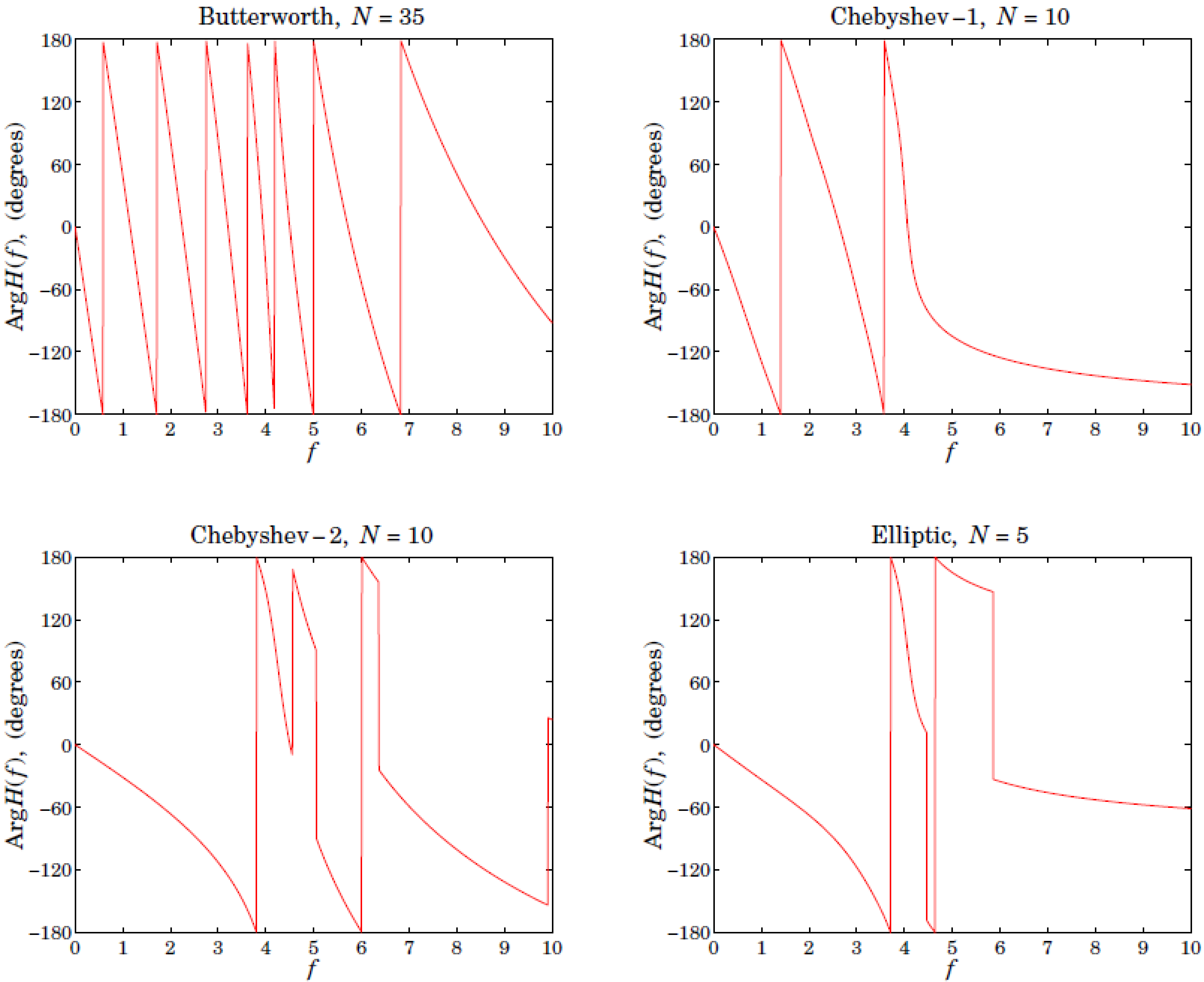
$$F_N(w) = \begin{cases} w^N\,, & \text{Butterworth} \\ C_N(w)\,, & \text{Chebyshev, type-1} \\ \left[k_1 C_N(k^{-1} w^{-1})\right]^{-1}\,, & \text{Chebyshev, type-2} \\ \mathrm{cd}(NuK_1, k_1)\,, \quad w = \mathrm{cd}(uK, k)\,, & \text{Elliptic} \end{cases}$$

$$H(s)H^*(-s)\Big|_{s=j\Omega} = |H(\Omega)|^2$$

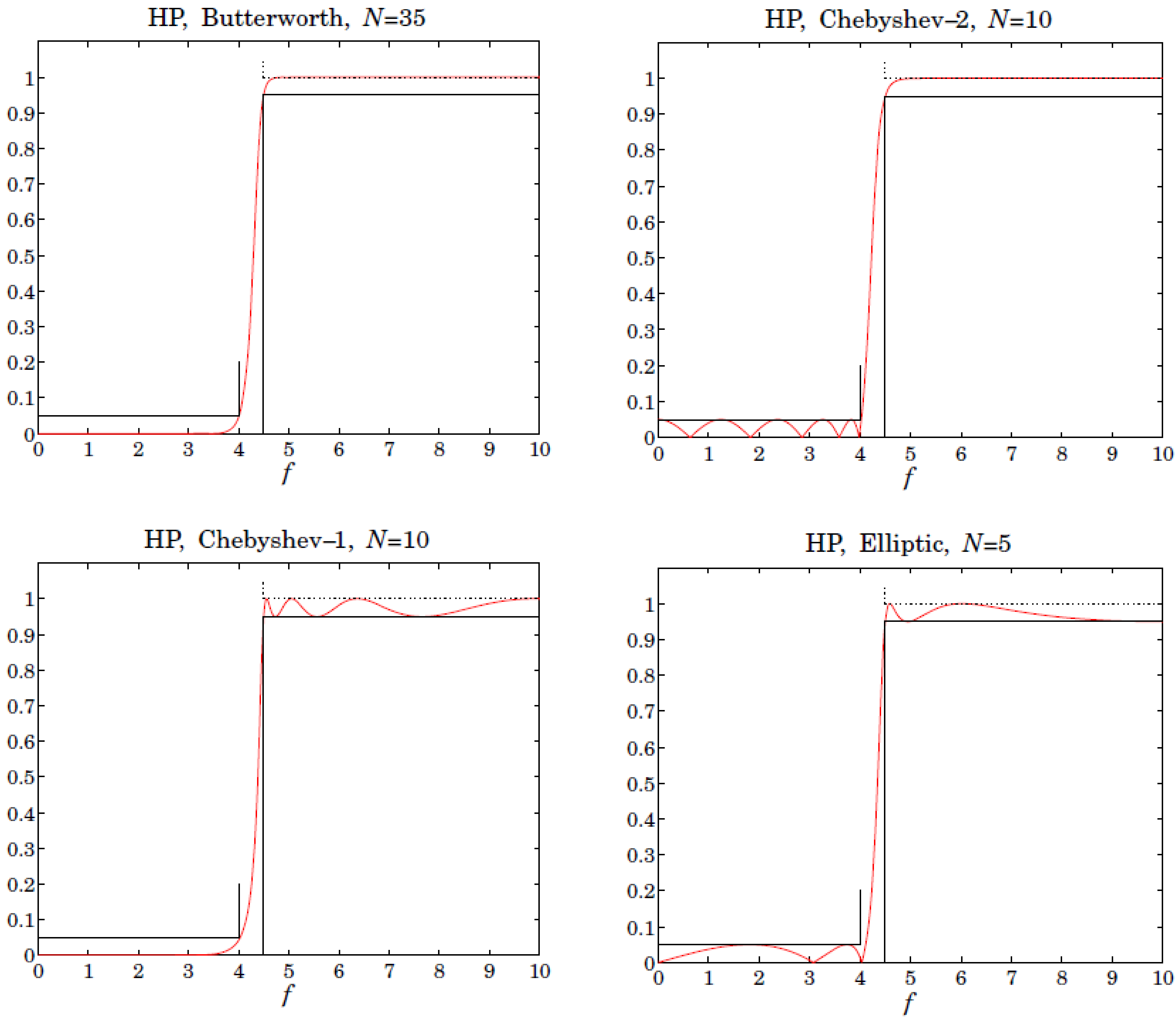
spectral factorization problem
has unique solution by requiring
stability & causality

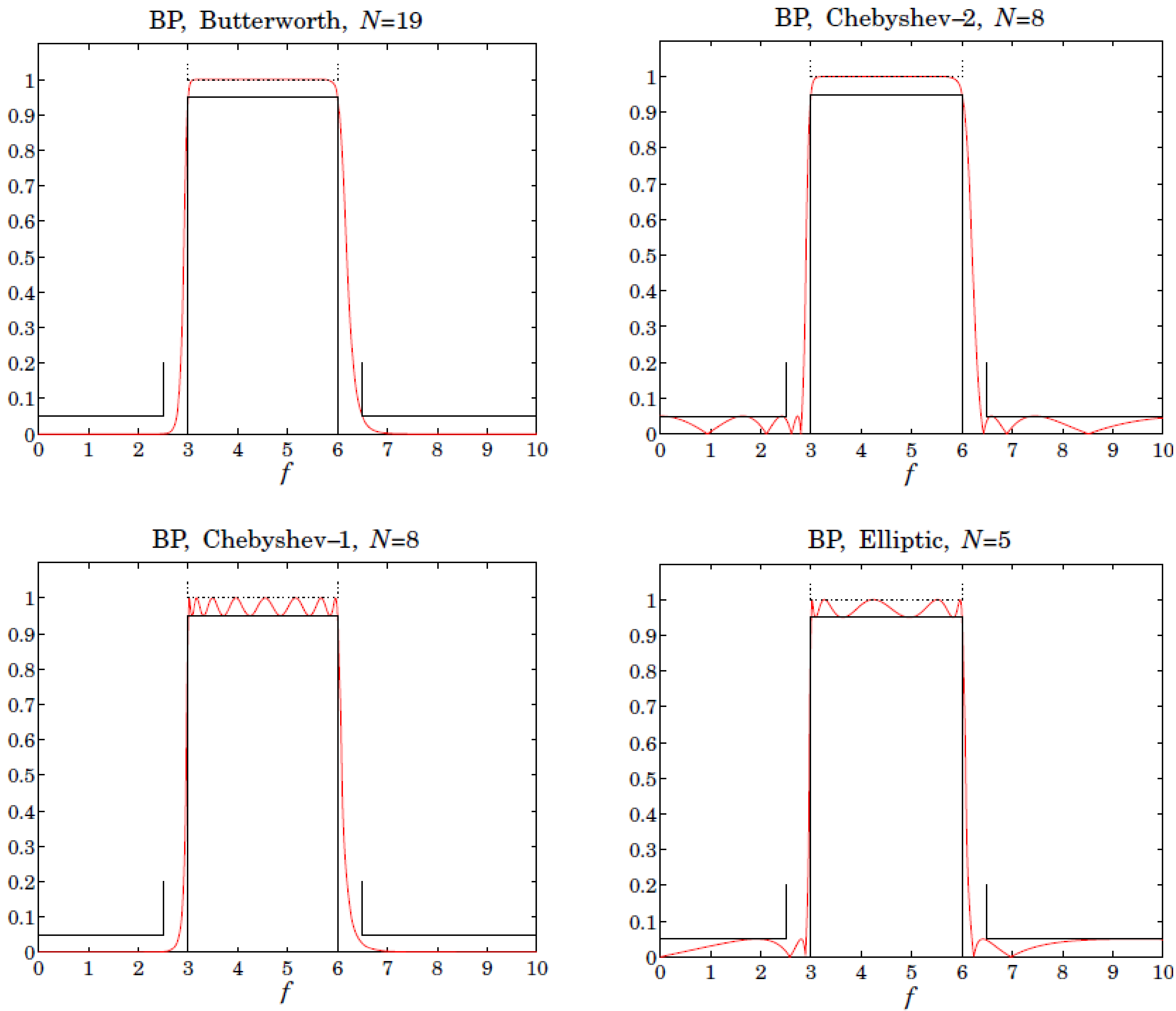


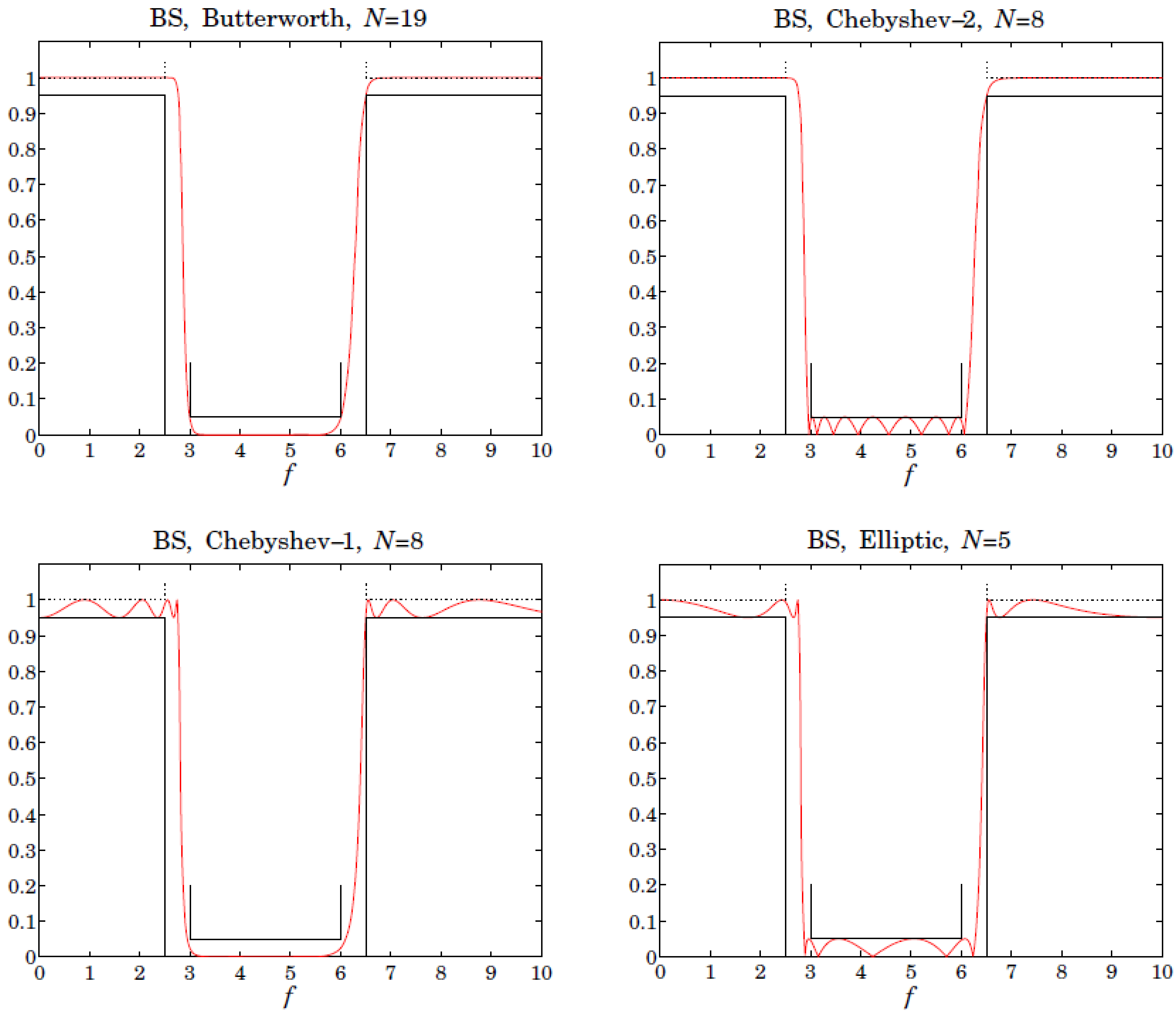
Butterworth, N = 35
Chebyshev−1, N = 10
Chebyshev−2, N = 10
Elliptic, N = 5
|H(f)|
f
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0 1 2 3 4 5 6 7 8 9 10



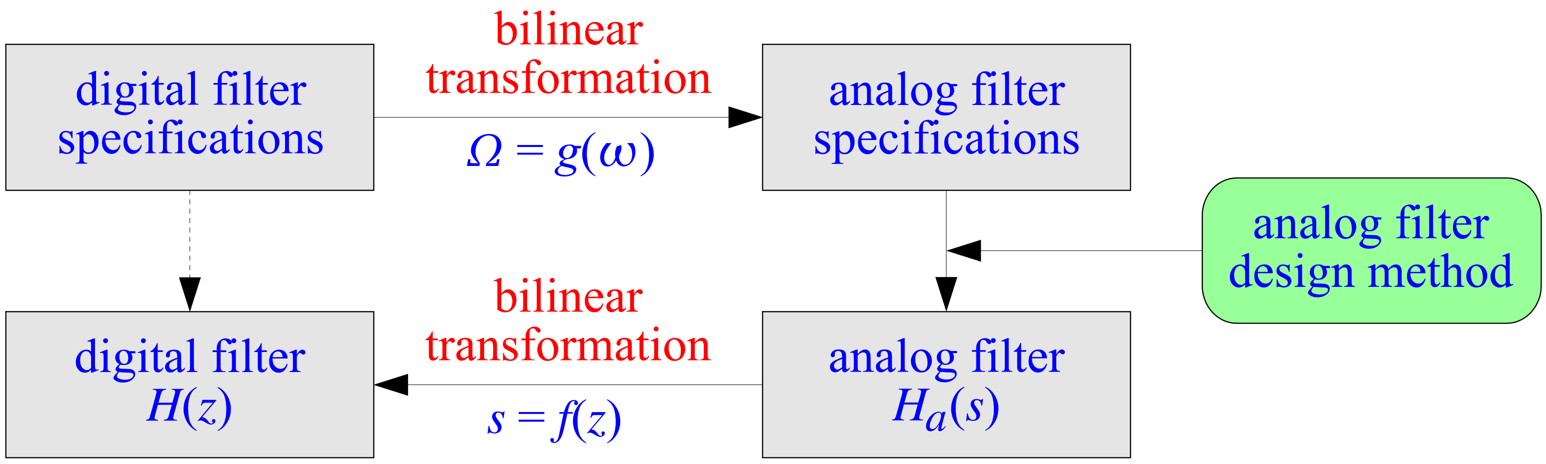
Butterworth, $N = 35$
Chebyshev−1, $N = 10$
Chebyshev−2, $N = 10$
Elliptic, $N = 5$
$\mathrm{Arg}H(f)$, (degrees)
f
180
120
60
0
−60
−120
−180
0 1 2 3 4 5 6 7 8 9 10


HP, Butterworth, N=35
HP, Chebyshev–2, N=10
HP, Chebyshev–1, N=10
HP, Elliptic, N=5
f


BP, Butterworth, N=19
BP, Chebyshev–2, N=8
BP, Chebyshev–1, N=8
BP, Elliptic, N=5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 1 2 3 4 5 6 7 8 9 10
f


BS, Butterworth, N=19
BS, Chebyshev–2, N=8
BS, Chebyshev–1, N=8
BS, Elliptic, N=5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 1 2 3 4 5 6 7 8 9 10
f

summary – digital designs



all cases are implemented by the following MATLAB functions, from **notes.pdf**

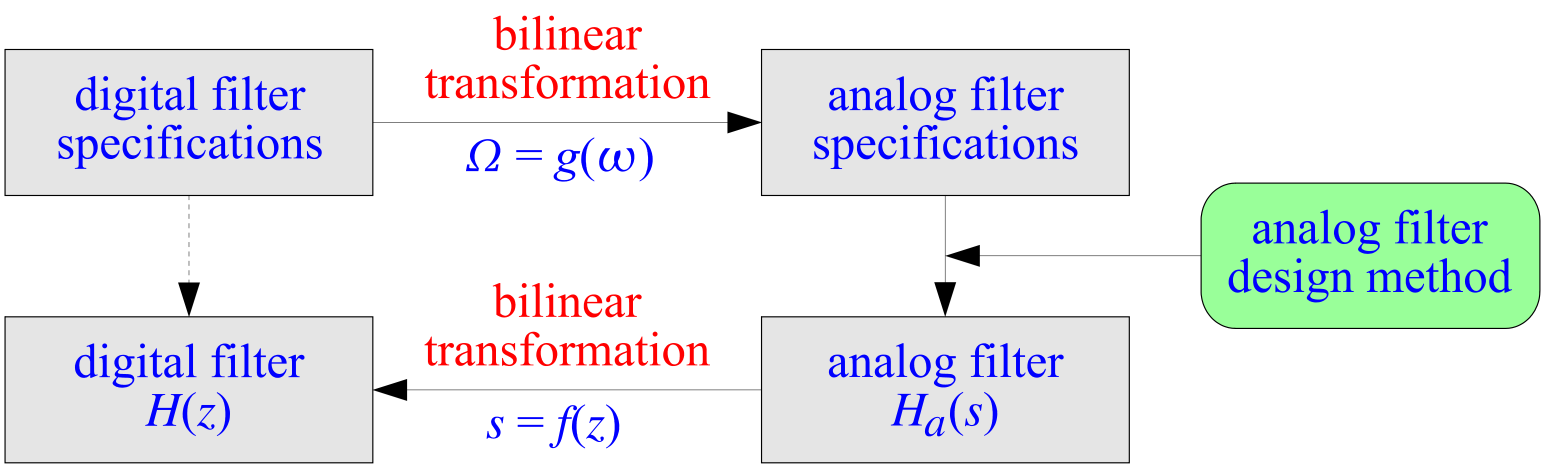
```
% dford.m  -  digital filter order determination
% Usage: [N,Ad,wd] = dford(wp,ws,Ap,As,type,match);

% dfdes.m  - digital filter design with bilinear transformation
% Usage: [B,A,w0] = dfdes(N, Ad, wd, type, shape, coeffs);
```

IIR filter design – bilinear transformation

One of the simplest and effective methods of designing IIR digital filters with prescribed magnitude response specifications is the *bilinear transformation* method.

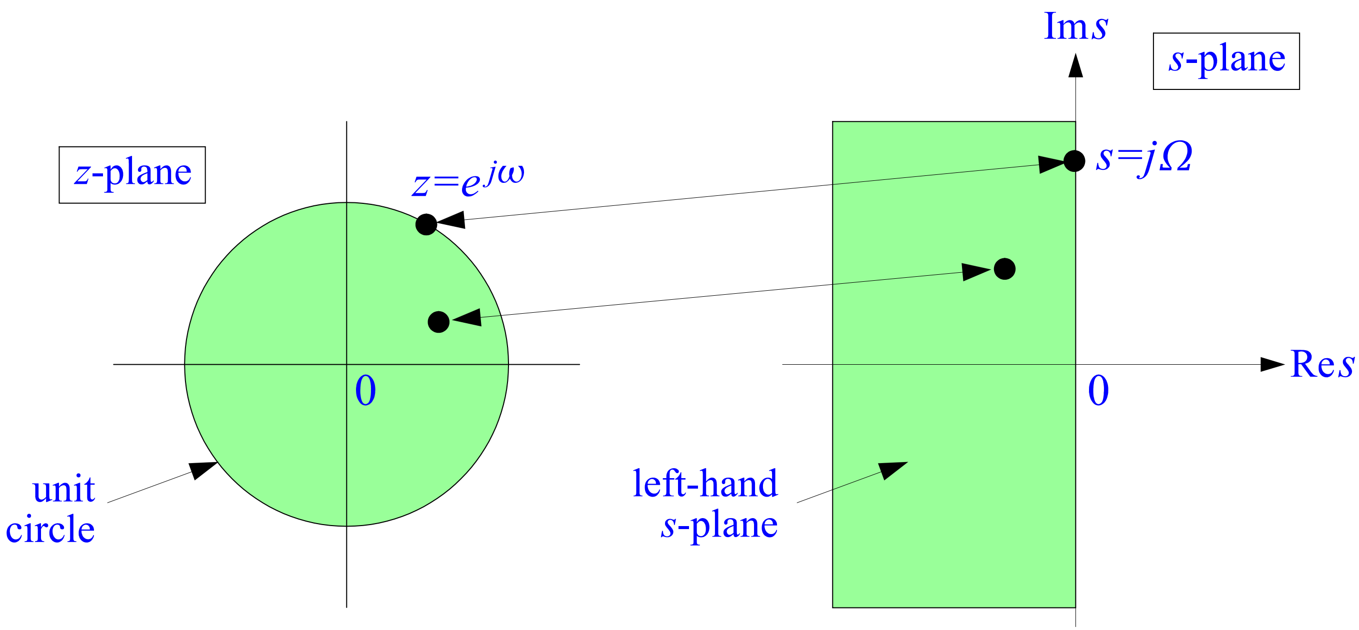
Instead of designing the digital filter directly, the method maps the digital filter into a fictitious *equivalent analog* filter, which can be designed by one of the well-developed analog filter design methods, such as Butterworth, Chebyshev, or elliptic filter designs. The designed analog filter is then mapped back into the desired digital filter by the blinear transformation. The procedure is illustrated below.



The mapping between the s and z planes is carried out by a transformation of the form:

$$\boxed{s = f(z)}$$

The mapping $f(z)$ is chosen so that it maps the left-hand s-plane into the inside of the unit circle on the z-plane, as shown below.



Because all analog filter design methods give rise to stable and causal transfer functions $H_a(s)$, this property guarantees that the designed digital filter $H(z)$ will also be stable and causal.

In addition, the transformation must map the s-plane frequency axis, that is, the imaginary axis $s = j\Omega$ onto the z-plane frequency axis, that is, the periphery of the unit circle $z = e^{j\omega}$, so that,

$$j\Omega = f(e^{j\omega}) \quad \Rightarrow \quad \boxed{\Omega = g(\omega)}$$

The overall design method can be summarized as follows: Starting with given magnitude response specifications for the digital filter, the specifications are transformed by the *appropriate* prewarping transformation, $\Omega = g(\omega)$, into the specifications of an equivalent analog filter.

Using an analog filter design technique, the equivalent analog filter, say $H_a(s)$, is designed. Using the bilinear transformation, $s = f(z)$, the analog filter is mapped back into the desired digital filter $H(z)$, by defining:

$$\boxed{H(z) = H_a(s)\Big|_{s=f(z)} = H_a\big(f(z)\big)}$$

The corresponding frequency responses also map in a similar fashion:

$$\boxed{H(\omega) = H_a(\Omega)\Big|_{\Omega=g(\omega)} = H_a\big(g(\omega)\big)}$$

There are several types of bilinear transformations, depending on the desired type filter to be designed, lowpass, highpass, bandpass, bandstop. They are defined as follows:

(lowpass)	$s = f(z) = \dfrac{1 - z^{-1}}{1 + z^{-1}}$
(highpass)	$s = f(z) = \dfrac{1 + z^{-1}}{1 - z^{-1}}$
(bandpass)	$s = f(z) = \dfrac{1 - 2cz^{-1} + z^{-2}}{1 - z^{-2}}$
(bandstop)	$s = f(z) = \dfrac{1 - z^{-2}}{1 - 2c + z^{-2}}$

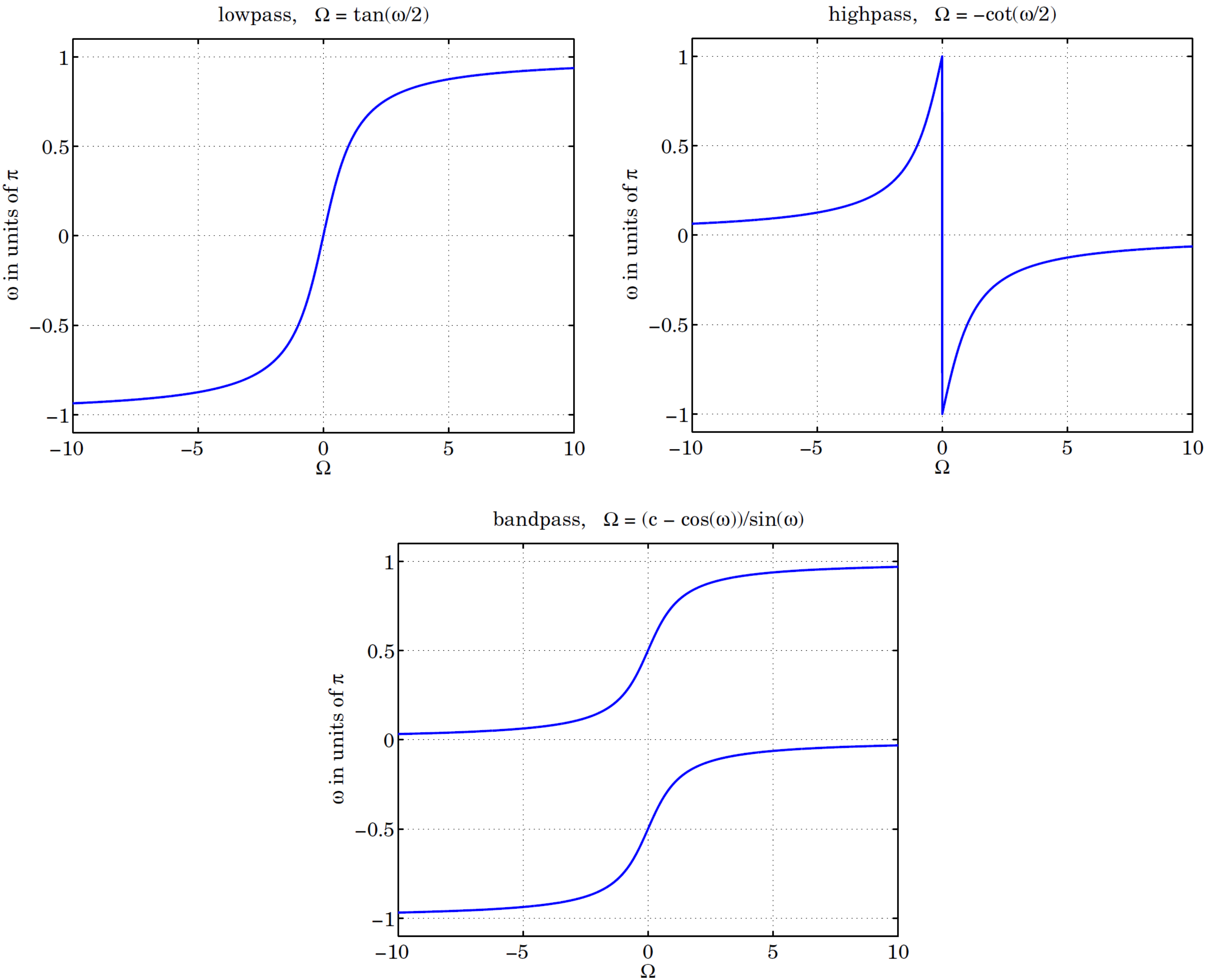
where, $c = \cos(\omega_0)$, with ω_0 the center of the passband band. Only the first two are "bilinear" in the variable z, the last two being quadrilinear.

All of the above transformations satisfy the constraint that they map the left-hand s-plane into the inside of the unit-circle on the z-plane.

the corresponding frequency maps are:

$$\boxed{\begin{array}{ll} \text{(lowpass)} & \Omega = g(\omega) = \tan\left(\dfrac{\omega}{2}\right) \\[2ex] \text{(highpass)} & \Omega = g(\omega) = -\cot\left(\dfrac{\omega}{2}\right) \\[2ex] \text{(bandpass)} & \Omega = g(\omega) = \dfrac{c - \cos\omega}{\sin\omega} \\[2ex] \text{(bandstop)} & \Omega = g(\omega) = \dfrac{\sin\omega}{\cos\omega - c} \end{array}}$$

Because of the nonlinear relationship between the physical frequency ω and the fictitious analog frequency Ω, such transformations are sometimes referred to as *frequency prewarping* transformations.



lowpass, Ω = tan(ω/2)
highpass, Ω = −cot(ω/2)
bandpass, Ω = (c − cos(ω))/sin(ω)
ω in units of π
Ω
1
0.5
0
−0.5
−1
−10
−5
0
5
10

For example, for the lowpass type, we have

$$s = f(z) = \frac{1 - z^{-1}}{1 + z^{-1}}$$

The corresponding mapping of frequencies is obtained as follows:

$$j\Omega = f(e^{j\omega}) = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}} = j\,\frac{\sin(\omega/2)}{\cos(\omega/2)} = j\tan\left(\frac{\omega}{2}\right)$$

or,

$$\boxed{\Omega = g(\omega) = \tan\left(\frac{\omega}{2}\right)}$$

We also have,

$$\mathrm{Re}\, s = \frac{1}{2}(s+s^*) = \frac{1}{2}\left[\frac{z-1}{z+1} + \frac{z^*-1}{z^*+1}\right] = \frac{(z-1)(z^*+1) + (z+1)(z^*-1)}{2(z+1)(z^*+1)}$$

or,

$$\mathrm{Re}\, s = \frac{|z|^2 - 1}{|z+1|^2}$$

which shows that

$$\mathrm{Re}\, s < 0 \quad \Leftrightarrow \quad |z| < 1 \qquad \text{and} \qquad \mathrm{Re}\, s = 0 \quad \Leftrightarrow \quad |z| = 1$$

There are two approaches to using these transformations in designing digital filters:

(a) Given a desired digital filter type, LP, HP, BP, BS, one uses the lowpass bilinear transformation to map to an analog filter of a similar type, LP, HP, BP, BS. This approach is used, for example in designing 2nd order parameter audio equalizer filters.

(b) Given a desired digital filter type, LP, HP, BP, BS, one uses the corresponding LP, HP, BP, BS transformation type to map to a lowpass analog prototype filter, which is the transformed back to tha appropriate digital filter type. Because it is easier to design lowpass analog prototypes, this approach is preferred in designing high-order digital filters.

approach (a):

$$\boxed{\begin{array}{lcl} \text{LP digital} & \xrightarrow{\text{LP}} & \text{LP analog} \\ \text{HP digital} & \xrightarrow{\text{LP}} & \text{HP analog} \\ \text{BP digital} & \xrightarrow{\text{LP}} & \text{BP analog} \\ \text{BS digital} & \xrightarrow{\text{LP}} & \text{BS analog} \end{array}}$$

approach (b):

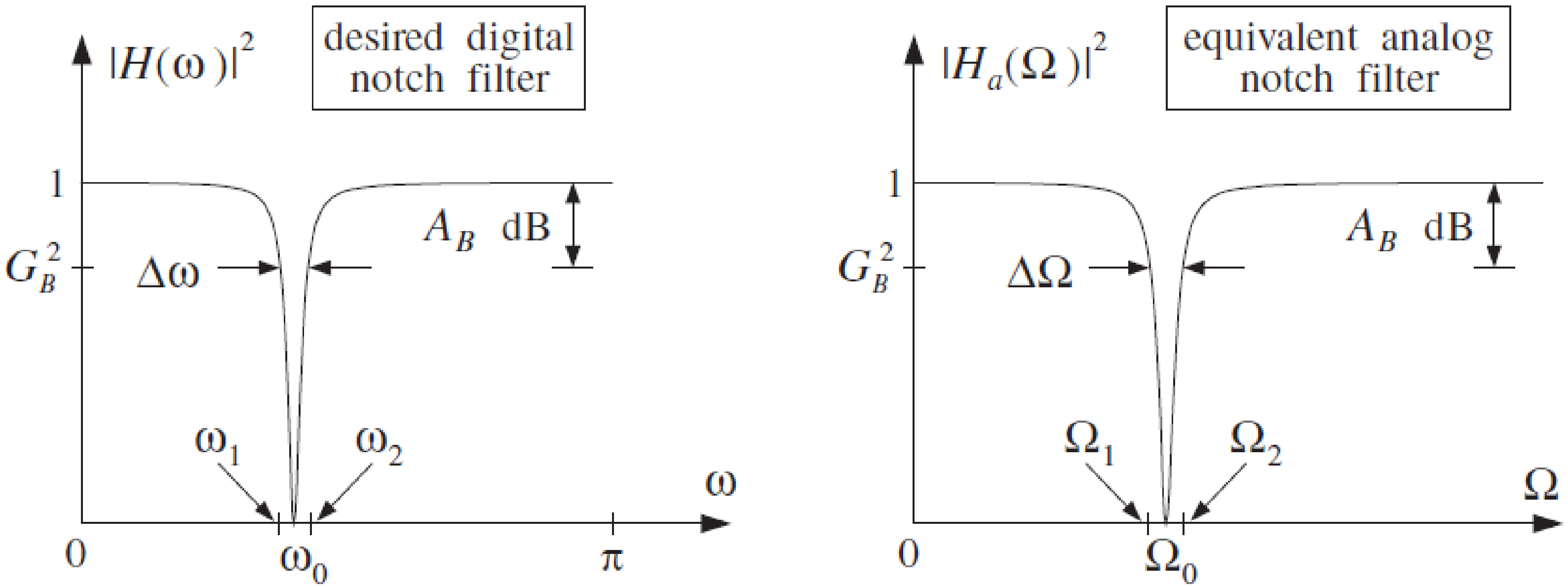
$$\boxed{\begin{array}{lcl} \text{LP digital} & \xrightarrow{\text{LP}} & \text{LP analog} \\ \text{HP digital} & \xrightarrow{\text{HP}} & \text{LP analog} \\ \text{BP digital} & \xrightarrow{\text{BP}} & \text{LP analog} \\ \text{BS digital} & \xrightarrow{\text{BS}} & \text{LP analog} \end{array}}$$

Next, we apply approach (a) to design notch, peaking, and equalizer filters with exact specifications of center frequencies and bandwidths, as opposed to the apptoximate pole/zero designns that we considered earlier.

For narrow-width filters the pole/zero technique is adequate, but it becomes cumbersome for wider peak widths, such as those that might be used in graphic and parametric audio equalizers. The bilinear transformation method offers precise control over the desired specifications of such filters.

Notch Filters

notch digital maps to notch analog



The desired specifications are the sampling rate f_s, notch frequency f_0, and bandwidth Δf of the notch, or, equivalently, the corresponding digital frequencies:

$$\boxed{\omega_0 = \frac{2\pi f_0}{f_s}, \quad \Delta\omega = \frac{2\pi \Delta f}{f_s}}$$

Alternatively, we may specify ω_0 and the Q-factor,

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f}$$

The specifications together with their bilinear analog equivalents are shown above. The bandwidth $\Delta\omega$ is usually defined to be the 3-dB width, that is, the *full width at half maximum* of the magnitude squared response. More generally, it can be defined to be the full width at a level G_B^2, or in decibels:

$$A_B = -10\log_{10}(G_B^2) \qquad \Rightarrow \qquad G_B = 10^{-A_B/20}$$

The bandwidth $\Delta\omega$ is defined as the difference $\Delta\omega = \omega_2 - \omega_1$ of the left and right bandwidth frequencies ω_1 and ω_2 that are solutions of the equation, $|H(\omega)|^2 = G_B^2$. For the 3-dB width, we have the condition $|H(\omega)|^2 = 1/2$.

Given the desired specifications, $\{\omega_0, \Delta\omega, G_B^2\}$, the design procedure begins with the following expression for the equivalent analog filter, which has a notch at frequency $\Omega = \Omega_0$:

$$\boxed{H_a(s) = \frac{s^2 + \Omega_0^2}{s^2 + \alpha s + \Omega_0^2}}$$

We will see below that the filter parameters $\{\alpha, \Omega_0\}$ can be calculated from the given specifications by the following design equations:

$$\boxed{\Omega_0 = \tan\left(\frac{\omega_0}{2}\right), \quad \alpha = \frac{\sqrt{1 - G_B^2}}{G_B}(1 + \Omega_0^2)\tan\left(\frac{\Delta\omega}{2}\right)}$$

Then, using the bilinear transformation $s = (1 - z^{-1})/(1 + z^{-1})$, the filter $H_a(s)$ is transformed into the digital filter $H(z)$ as follows:

$$H(z) = H_a(s) = \frac{s^2 + \Omega_0^2}{s^2 + \alpha s + \Omega_0^2} = \frac{\left(\dfrac{1 - z^{-1}}{1 + z^{-1}}\right)^2 + \Omega_0^2}{\left(\dfrac{1 - z^{-1}}{1 + z^{-1}}\right)^2 + \alpha\left(\dfrac{1 - z^{-1}}{1 + z^{-1}}\right) + \Omega_0^2}$$

$$= \frac{(1 - z^{-1})^2 + \Omega_0^2(1 + z^{-1})^2}{(1 - z^{-1})^2 + \alpha(1 - z^{-1})(1 + z^{-1}) + \Omega_0^2(1 + z^{-1})^2}$$

$$= \left(\frac{1 + \Omega_0^2}{1 + \Omega_0^2 + \alpha}\right) \frac{1 - 2\left(\dfrac{1 - \Omega_0^2}{1 + \Omega_0^2}\right) z^{-1} + z^{-2}}{1 - 2\left(\dfrac{1 - \Omega_0^2}{1 + \Omega_0^2 + \alpha}\right) z^{-1} + \left(\dfrac{1 + \Omega_0^2 - \alpha}{1 + \Omega_0^2 + \alpha}\right) z^{-2}}$$

The coefficients of the digital filter can be simplified considerably by recognizing that α already has a factor $(1+\Omega_0^2)$ in its definition. Thus, we may replace it by

$$\alpha = (1+\Omega_0^2)\beta$$

where

$$\boxed{\beta = \frac{\sqrt{1-G_B^2}}{G_B}\tan\left(\frac{\Delta\omega}{2}\right)}$$

Using some trigonometry, we can write also

$$\frac{1-\Omega_0^2}{1+\Omega_0^2} = \frac{1-\tan^2(\omega_0/2)}{1+\tan^2(\omega_0/2)} = \cos\omega_0$$

Canceling several common factors of $(1+\Omega_0^2)$, we can write the transfer function $H(z)$ in the simplified form:

$$\boxed{H(z) = \left(\frac{1}{1+\beta}\right)\frac{1-2\cos\omega_0\, z^{-1}+z^{-2}}{1-2\left(\frac{\cos\omega_0}{1+\beta}\right)z^{-1}+\left(\frac{1-\beta}{1+\beta}\right)z^{-2}}}$$

This is the final design. It expresses the filter coefficients in terms of the design specifications $\{\omega_0, \Delta\omega, G_B^2\}$. Note that the numerator has a notch at the desired frequency ω_o and its conjugate $-\omega_0$, because it factors into:

$$1 - 2\cos\omega_0\, z^{-1} + z^{-2} = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$$

The first one, $\Omega_0 = \tan(\omega_0/2)$, is simply the bilinear transformation of ω_0 and makes the analog filter's notch correspond to the digital filter's notch. The equation for α can be derived as follows. Setting $s = j\Omega$ in $H_a(s)$, we obtain the frequency and magnitude responses:

$$H_a(\Omega) = \frac{-\Omega^2 + \Omega_0^2}{-\Omega^2 + j\alpha\Omega + \Omega_0^2} \quad \Rightarrow \quad |H_a(\Omega)|^2 = \frac{(\Omega^2 - \Omega_0^2)^2}{(\Omega^2 - \Omega_0^2)^2 + \alpha^2\Omega^2}$$

It is evident from these expressions that $H_a(\Omega)$ has a notch at $\Omega = \pm\Omega_0$.

The analog bandwidth frequencies Ω_1 and Ω_2 are solutions of the equation $|H_a(\Omega)|^2 = G_B^2$, that is,

$$\frac{(\Omega^2 - \Omega_0^2)^2}{(\Omega^2 - \Omega_0^2)^2 + \alpha^2\Omega^2} = G_B^2$$

Eliminating the denominator and rearranging terms, we can write it as the quartic equation in Ω:

$$\Omega^4 - \left(2\Omega_0^2 + \frac{G_B^2}{1 - G_B^2}\alpha^2\right)\Omega^2 + \Omega_0^4 = 0$$

It may be thought of as a quadratic equation in the variable $x = \Omega^2$, that is,

$$x^2 - \left(2\Omega_0^2 + \frac{G_B^2}{1 - G_B^2}\alpha^2\right)x + \Omega_0^4 = 0$$

Let $x_1 = \Omega_1^2$ and $x_2 = \Omega_2^2$ be its two solutions. Rather than solving it, we use the properties that the sum and product of the two solutions are related to the first and second coefficients of the quadratic by:

$$\Omega_1^2 + \Omega_2^2 = x_1 + x_2 = 2\Omega_0^2 + \frac{G_B^2}{1 - G_B^2}\alpha^2$$

$$\Omega_1^2\Omega_2^2 = x_1x_2 = \Omega_0^4$$

From the second equation, we obtain:

$$\boxed{\Omega_1\Omega_2 = \Omega_0^2}$$

which states that Ω_0 is the *geometric mean* of the left and right bandwidth frequencies. Using this result, we obtain:

$$\Omega_1^2 + \Omega_2^2 = 2\Omega_1\Omega_2 + \frac{G_B^2}{1-G_B^2}\alpha^2$$

which allows us to solve for the analog bandwidth:

$$\Delta\Omega^2 = (\Omega_2 - \Omega_1)^2 = \Omega_1^2 + \Omega_2^2 - 2\Omega_1\Omega_2 = \frac{G_B^2}{1-G_B^2}\alpha^2$$

or,

$$\boxed{\Delta\Omega = \Omega_2 - \Omega_1 = \frac{G_B}{\sqrt{1-G_B^2}}\,\alpha} \qquad (A_B\text{-dB width})$$

Solving for α, we have:

$$\boxed{\alpha = \frac{\sqrt{1-G_B^2}}{G_B}\,\Delta\Omega}$$

Note that for the 3-dB case, $G_B^2 = 1/2$, the parameter α is equal to the 3-dB bandwidth:

$$\boxed{\alpha = \Delta\Omega} \qquad \text{(analog 3-dB width)}$$

Finally, we must relate the analog bandwidth $\Delta\Omega$ to the physical bandwidth $\Delta\omega = \omega_2 - \omega_1$. Using the bilinear transformations $\Omega_1 = \tan(\omega_1/2)$, $\Omega_2 = \tan(\omega_2/2)$, and some trigonometry, we find:

$$\begin{aligned}
\tan\left(\frac{\Delta\omega}{2}\right) = \tan\left(\frac{\omega_2-\omega_1}{2}\right) &= \frac{\tan(\omega_2/2) - \tan(\omega_1/2)}{1+\tan(\omega_2/2)\tan(\omega_1/2)} \\
&= \frac{\Omega_2 - \Omega_1}{1+\Omega_2\Omega_1} = \frac{\Delta\Omega}{1+\Omega_0^2}
\end{aligned}$$

where we used $\Omega_1\Omega_2 = \Omega_0^2$. Solving for $\Delta\Omega$, we have:

$$\Delta\Omega = (1+\Omega_0^2)\tan\left(\frac{\Delta\omega}{2}\right)$$

Thus, finally,

$$\boxed{\alpha = \frac{\sqrt{1-G_B^2}}{G_B}\,\Delta\Omega = \frac{\sqrt{1-G_B^2}}{G_B}\,(1+\Omega_0^2)\tan\left(\frac{\Delta\omega}{2}\right)}$$

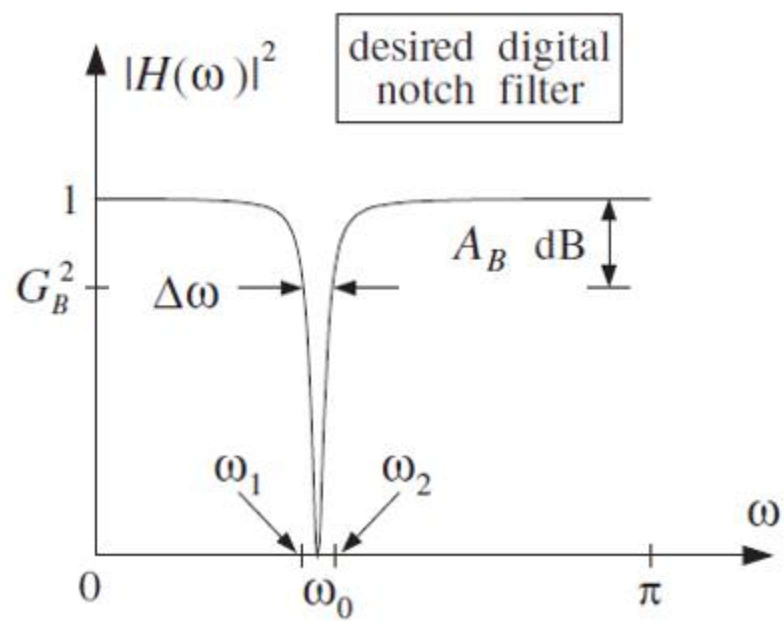
To summarize, we have the exact design equations,

$$H(z) = \left(\frac{1}{1+\beta}\right) \frac{1 - 2\cos\omega_0\, z^{-1} + z^{-2}}{1 - 2\left(\frac{\cos\omega_0}{1+\beta}\right) z^{-1} + \left(\frac{1-\beta}{1+\beta}\right) z^{-2}}$$

$$\beta = \frac{\sqrt{1-G_B^2}}{G_B} \tan\left(\frac{\Delta\omega}{2}\right)$$

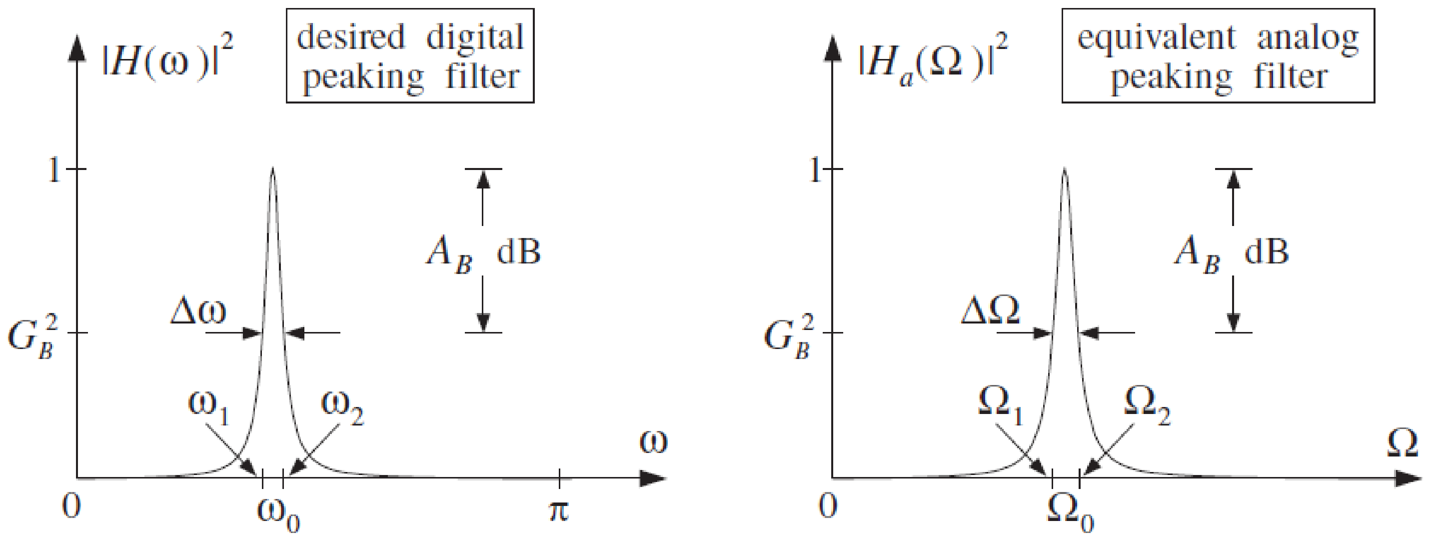
where $\Delta\omega$ is the G_B-level bandwith. In the 3-dB width case, we have

$$\beta = \tan\left(\frac{\Delta\omega}{2}\right)$$



Peaking filters

peaking digital maps to peaking analog



Peaking or resonator filters can be designed in a similar fashion. The desired specifications are shown above. The design procedure starts with the second-order analog resonator filter:

$$\boxed{H_a(s) = \frac{\alpha s}{s^2 + \alpha s + \Omega_0^2}}$$

which has frequency and magnitude responses:

$$H_a(\Omega) = \frac{j\alpha\Omega}{-\Omega^2 + j\alpha\Omega + \Omega_0^2} \quad \Rightarrow \quad |H_a(\Omega)|^2 = \frac{\alpha^2\Omega^2}{(\Omega^2 - \Omega_0^2)^2 + \alpha^2\Omega^2}$$

Note that $H_a(\Omega)$ has unity gain at the peak frequencies $\Omega = \pm\Omega_0$.

The bandwidth frequencies Ω_1 and Ω_2 will satisfy the bandwidth condition:

$$|H_a(\Omega)|^2 = \frac{\alpha^2\Omega^2}{(\Omega^2 - \Omega_0^2)^2 + \alpha^2\Omega^2} = G_B^2$$

It can be written as the quartic:

$$\Omega^4 - \left(2\Omega_0^2 + \frac{1 - G_B^2}{G_B^2}\alpha^2\right)\Omega^2 + \Omega_0^4 = 0$$

which is similar to the notch case. Its two solutions Ω_1^2 and Ω_2^2 satisfy the conditions:

$$\Omega_1^2 + \Omega_2^2 = 2\Omega_0^2 + \frac{1 - G_B^2}{G_B^2}\alpha^2$$

$$\Omega_1^2\Omega_2^2 = \Omega_0^4$$

from which we obtain $\Omega_1\Omega_2 = \Omega_0^2$ and

$$\Delta\Omega = \Omega_2 - \Omega_1 = \frac{\sqrt{1 - G_B^2}}{G_B}\alpha \quad \Rightarrow \quad \boxed{\alpha = \frac{G_B}{\sqrt{1 - G_B^2}}\Delta\Omega}$$

The analog filter parameters $\{\alpha, \Omega_0\}$ are given by equations similar to the notch case:

$$\boxed{\Omega_0 = \tan\left(\frac{\omega_0}{2}\right), \quad \alpha = \frac{G_B}{\sqrt{1-G_B^2}}(1+\Omega_0^2)\tan\left(\frac{\Delta\omega}{2}\right)}$$

The digital filter is obtained by the bilinear transformation:

$$H(z) = H_a(s) = \left.\frac{\alpha s}{s^2 + \alpha s + \Omega_0^2}\right|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

The final design equations are:

$$H(z) = \left(\frac{\beta}{1+\beta}\right) \frac{1 - z^{-2}}{1 - 2\left(\frac{\cos\omega_0}{1+\beta}\right) z^{-1} + \left(\frac{1-\beta}{1+\beta}\right) z^{-2}}$$

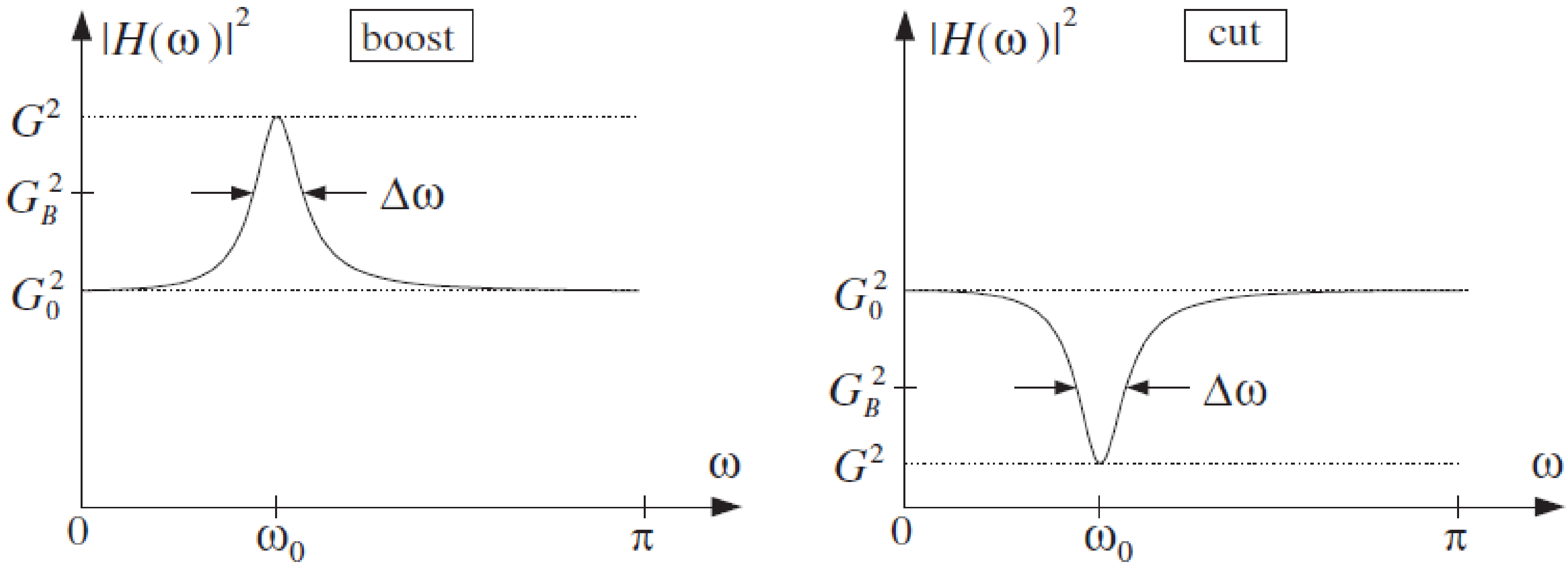
where β is now given by,

$$\beta = \frac{G_B}{\sqrt{1 - G_B^2}} \tan\left(\frac{\Delta\omega}{2}\right)$$

Note that the numerator vanishes at $z = \pm 1$, that is, at DC and the Nyquist frequency. For the 3-dB widths, we have $G_B^2 = 1/2$, and the parameters β is the same as in the notch filter, that is,

$$\beta = \tan\left(\frac{\Delta\omega}{2}\right)$$

Parametric Equalizer Filters



Equalization (EQ) is a very common operation in audio systems—analog, digital, home, car, public, or studio recording systems.

Graphic equalizers are the more common type, in which the audio band is divided into a *fixed* number of frequency bands, and the amount of equalization in each band is controlled by a bandpass filter whose gain can be varied up and down. The center frequencies of the bands and the filter 3-dB widths are fixed, and the user can vary only the overall gain in each band.

A more flexible equalizer type is the parametric equalizer, in which all three filter parameters—gain, center frequency, and bandwidth—can be varied. Cascading four or five such filters together can cover the entire audio band and achieve almost any desired equalization effect.

The filter design problem is to determine the filter's transfer function in terms of the specification parameters: $\{G_0, G, G_B, \omega_0, \Delta\omega\}$. We define the parametric equalizer filter as a *linear combination* of the notching and peaking filters discussed above:

$$H(z) = G_0 H_{\text{notch}}(z) + G H_{\text{peak}}(z)$$

At ω_0 the gain is G, because the notch filter vanishes and the peak filter has unity gain. Similarly, at DC and the Nyquist frequency, the gain is equal to the reference G_0, because the notch is unity and the peak vanishes. When, $G = G_0$, we have $H(z) = G_0$, that is, no equalization.

Substituting the expressions for the peaking and notch filters, we obtain:

$$H(z) = \frac{\left(\frac{G_0 + G\beta}{1+\beta}\right) - 2\left(\frac{G_0 \cos\omega_0}{1+\beta}\right) z^{-1} + \left(\frac{G_0 - G\beta}{1+\beta}\right) z^{-2}}{1 - 2\left(\frac{\cos\omega_0}{1+\beta}\right) z^{-1} + \left(\frac{1-\beta}{1+\beta}\right) z^{-2}}$$

The parameter β is now given by,

$$\beta = \sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}} \tan\left(\frac{\Delta\omega}{2}\right)$$

Note that the quantity under the square root is always positive for either a boost or a cut.

The design equations can be justified as follows. Starting with the same linear combination of the analog versions of the notching and peaking filters, we obtain the analog version of $H(z)$:

$$\boxed{H_a(s) = G_0 H_{\text{notch}}(s) + G H_{\text{peak}}(s) = \frac{G_0(s^2 + \Omega_0^2) + G\alpha s}{s^2 + \alpha s + \Omega_0^2}}$$

Then, the bandwidth condition $|H_a(\Omega)|^2 = G_B^2$ can be stated as:

$$|H_a(\Omega)|^2 = \frac{G_0^2(\Omega^2 - \Omega_0^2)^2 + G^2\alpha^2\Omega^2}{(\Omega^2 - \Omega_0^2)^2 + \alpha^2\Omega^2} = G_B^2$$

It can be cast as the quartic equation:

$$\Omega^4 - \left(2\Omega_0^2 + \frac{G^2 - G_B^2}{G_B^2 - G_0^2}\alpha^2\right)\Omega^2 + \Omega_0^4 = 0$$

Proceeding as before and using the geometric-mean property $\Omega_1\Omega_2 = \Omega_0^2$, we find the relationship between the parameter α and the analog bandwidth $\Delta\Omega = \Omega_2 - \Omega_1$:

$$\alpha = \sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}}\,\Delta\Omega = \sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}}\,(1 + \Omega_0^2)\tan\left(\frac{\Delta\omega}{2}\right) \equiv (1 + \Omega_0^2)\beta$$

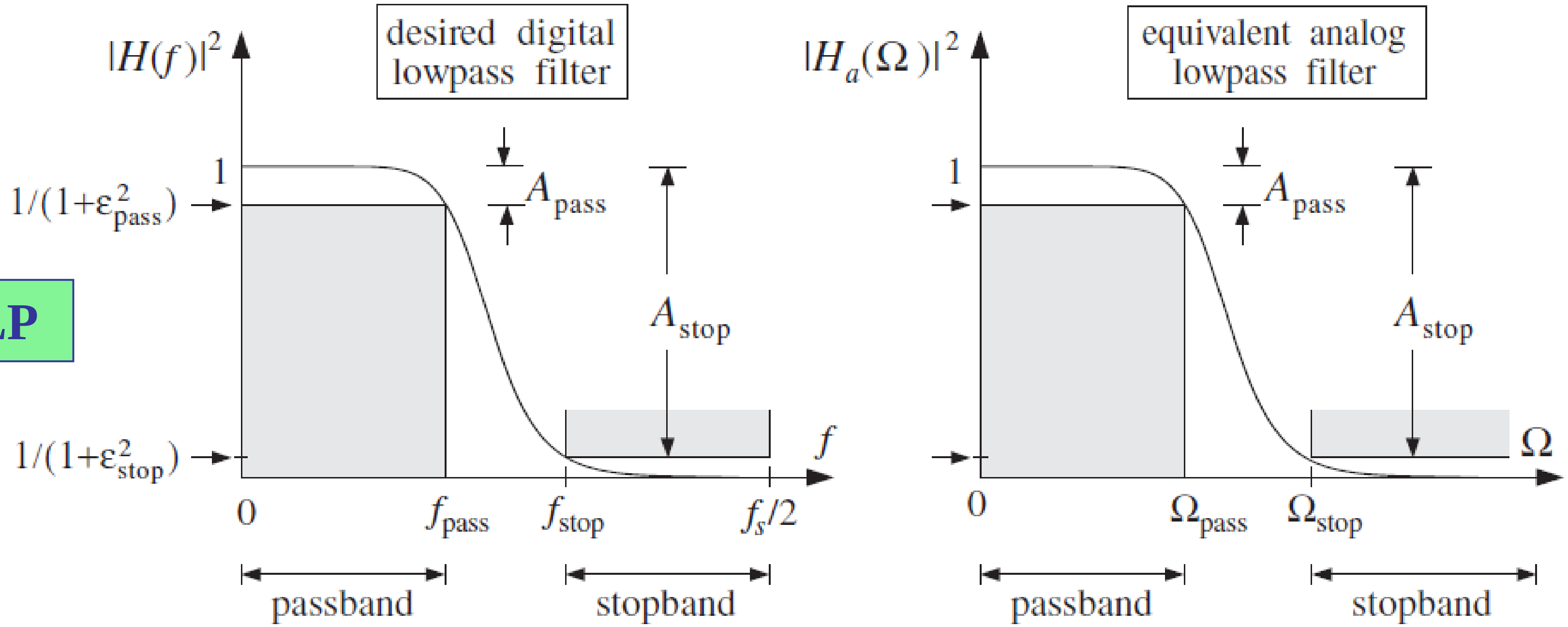
The design of notch, peaking, and EQ filters was based on approach (a) in which we map the same type of digital filter to a similar type of analog filter using the LP version of the bilinear transformation.

$$\text{approach (a):}\quad \boxed{\begin{array}{lcl} \text{LP digital} & \xrightarrow{\text{LP}} & \text{LP analog} \\ \text{HP digital} & \xrightarrow{\text{LP}} & \text{HP analog} \\ \text{BP digital} & \xrightarrow{\text{LP}} & \text{BP analog} \\ \text{BS digital} & \xrightarrow{\text{LP}} & \text{BS analog} \end{array}}$$

$$\text{approach (b):}\quad \boxed{\begin{array}{lcl} \text{LP digital} & \xrightarrow{\text{LP}} & \text{LP analog} \\ \text{HP digital} & \xrightarrow{\text{HP}} & \text{LP analog} \\ \text{BP digital} & \xrightarrow{\text{BP}} & \text{LP analog} \\ \text{BS digital} & \xrightarrow{\text{BS}} & \text{LP analog} \end{array}}$$

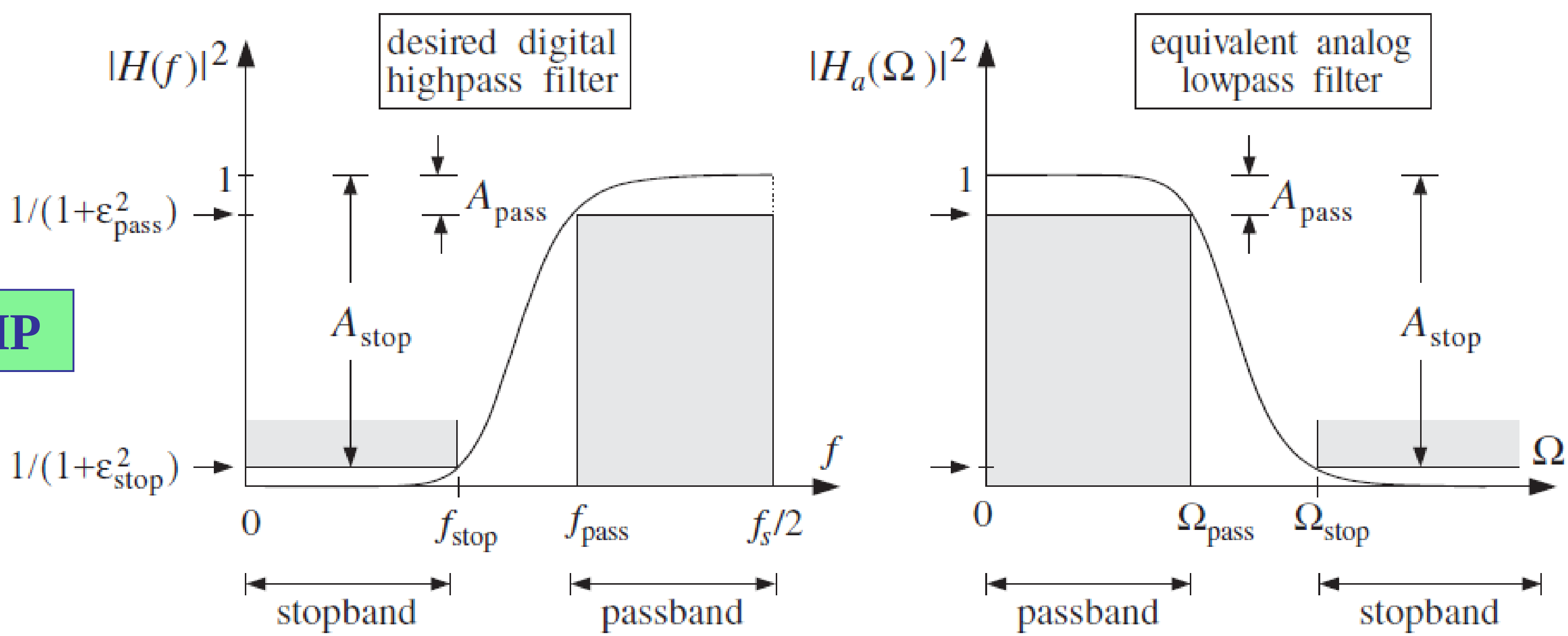
The design of higher-order filters is more conveniently based on approach (b) in which we always map the digital filter—regardless of its type—to a lowpass analog filter, but using the appropriate bilinear transformation type.

LP



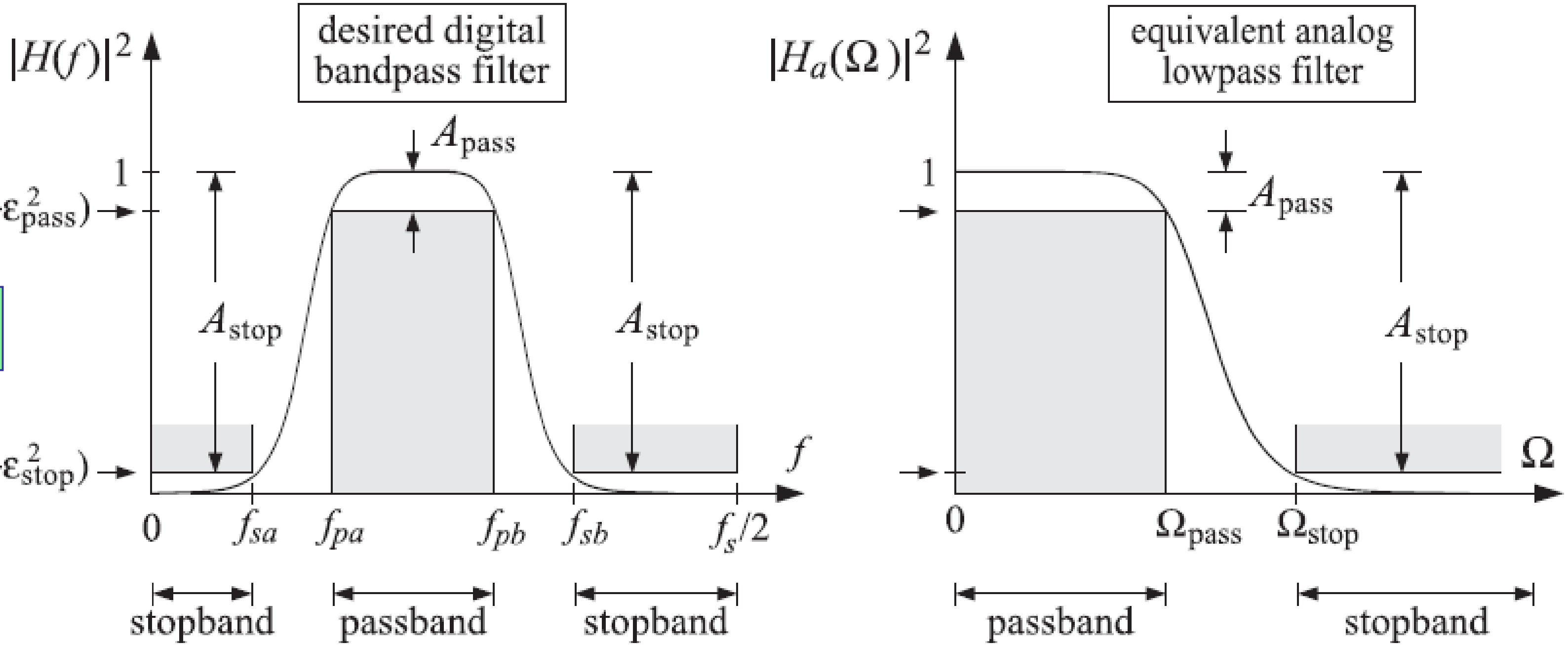
desired digital lowpass filter
$|H(f)|^2$
1
$1/(1+\varepsilon^2_{pass})$
$1/(1+\varepsilon^2_{stop})$
A_{pass}
A_{stop}
f
0
f_{pass}
f_{stop}
$f_s/2$
passband
stopband
equivalent analog lowpass filter
$|H_a(\Omega)|^2$
1
A_{pass}
A_{stop}
Ω
0
Ω_{pass}
Ω_{stop}
passband
stopband

HP

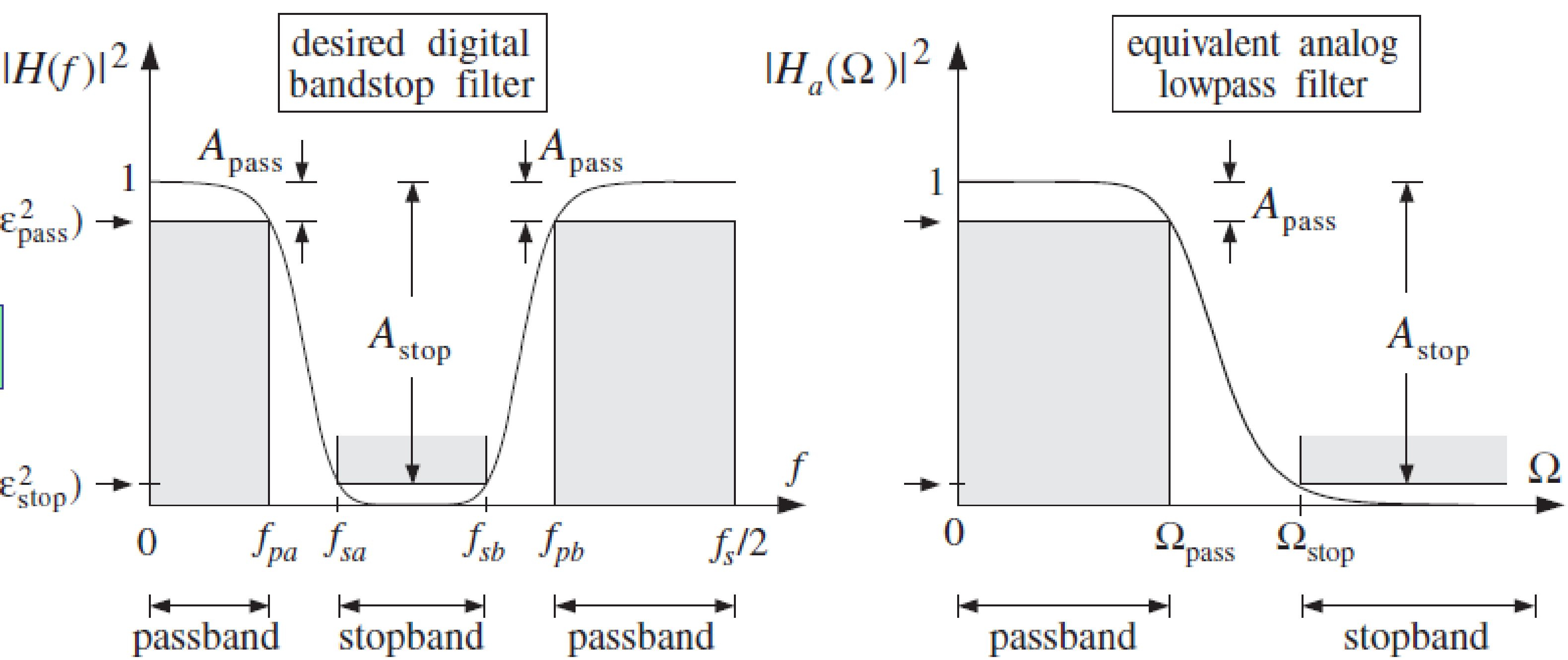


desired digital highpass filter
$|H(f)|^2$
1
$1/(1+\varepsilon^2_{pass})$
$1/(1+\varepsilon^2_{stop})$
A_{pass}
A_{stop}
f
0
f_{stop}
f_{pass}
$f_s/2$
stopband
passband
equivalent analog lowpass filter
$|H_a(\Omega)|^2$
1
A_{pass}
A_{stop}
Ω
0
Ω_{pass}
Ω_{stop}
passband
stopband

BP


desired digital bandpass filter
$|H(f)|^2$
1
$1/(1+\varepsilon^2_{pass})$
$1/(1+\varepsilon^2_{stop})$
A_{pass}
A_{stop}
A_{stop}
f
0
f_{sa}
f_{pa}
f_{pb}
f_{sb}
$f_s/2$
stopband
passband
stopband
equivalent analog lowpass filter
$|H_a(\Omega)|^2$
1
A_{pass}
A_{stop}
Ω
0
Ω_{pass}
Ω_{stop}
passband
stopband

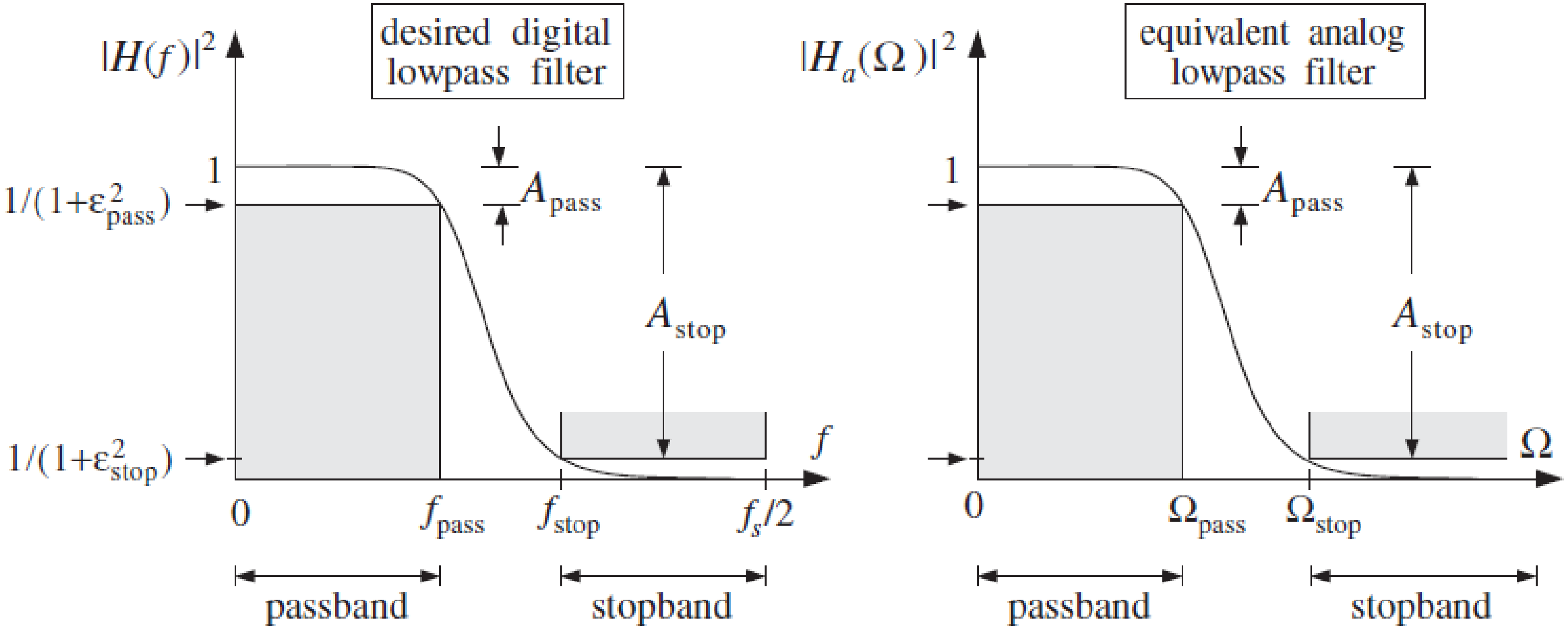
BS


desired digital bandstop filter
$|H(f)|^2$
1
$1/(1+\varepsilon^2_{pass})$
$1/(1+\varepsilon^2_{stop})$
A_{pass}
A_{pass}
A_{stop}
f
0
f_{pa}
f_{sa}
f_{sb}
f_{pb}
$f_s/2$
passband
stopband
passband
equivalent analog lowpass filter
$|H_a(\Omega)|^2$
1
A_{pass}
A_{stop}
Ω
0
Ω_{pass}
Ω_{stop}
passband
stopband

Higher-Order Filters

The second-order designs are adequate in some applications such as audio equalization, but are too limited when we need filters with very sharp cutoff specifications. Higher-order filters can achieve such sharp cutoffs, but at the price of increasing the filter complexity, that is, the filter order.

The figure below shows the specifications of a typical lowpass filter and its analog equivalent obtained by the bilinear transformation. The specification parameters are the four numbers $\{f_{\text{pass}}, f_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$, that is, the passband and stopband frequencies and the desired passband and stopband attenuations in dB.



Within the passband range $0 \le f \le f_{\text{pass}}$, the filter's attenuation is required to be *less* than A_{pass} decibels. And, within the stopband $f_{\text{stop}} \le f \le f_s/2$, it is required to be *greater* than A_{stop} decibels. Thus, the quantity A_{pass} is the *maximum* attenuation that can be tolerated in the passband and A_{stop} the *minimum* attenuation that must be achieved in the stopband.

The filter can be made into a *better* lowpass filter in three ways: (1) decreasing A_{pass} so that the passband becomes flatter, (2) increasing A_{stop} so that the stopband becomes deeper, and (3) moving f_{stop} closer to f_{pass} so that the *transition region* between passband and stopband becomes narrower. Thus, by appropriate choice of the specification parameters, the filter can be made as close to an ideal lowpass filter as desired.

Assuming the filter's magnitude response squared $|H(f)|^2$ is normalized to unity at DC, we can express the specification requirements as the following conditions on the filter's attenuation response in dB, defined as

$$A(f) = -10\log_{10}|H(f)|^2$$

$$\boxed{\begin{aligned} 0 \le A(f) &\le A_{\text{pass}}\,, \quad \text{for} \quad 0 \le f \le f_{\text{pass}} \\ A(f) &\ge A_{\text{stop}}\,, \quad \text{for} \quad f_{\text{stop}} \le f \le f_s/2 \end{aligned}}$$

Equivalently, in absolute units, the design specifications are:

$$\frac{1}{1+\varepsilon_{\text{pass}}^2} \le |H(f)|^2 \le 1\,, \quad \text{for} \quad 0 \le f \le f_{\text{pass}}$$

$$|H(f)|^2 \le \frac{1}{1+\varepsilon_{\text{stop}}^2}\,, \qquad \text{for} \quad f_{\text{stop}} \le f \le f_s/2$$

where $\{\varepsilon_{\text{pass}}, \varepsilon_{\text{stop}}\}$ are defined in terms of $\{A_{\text{pass}}, A_{\text{stop}}\}$ as follows:

$$
\begin{aligned}
|H(f_{\text{pass}})|^2 &= \frac{1}{1+\varepsilon_{\text{pass}}^2} = 10^{-A_{\text{pass}}/10}\,,\\
|H(f_{\text{stop}})|^2 &= \frac{1}{1+\varepsilon_{\text{stop}}^2} = 10^{-A_{\text{stop}}/10}\,,
\end{aligned}
$$

The quantities $\{\varepsilon_{\text{pass}}, \varepsilon_{\text{stop}}\}$ control the depths of the passband and stopband. They can be written in the equivalent forms:

$$
\boxed{\begin{aligned}
\varepsilon_{\text{pass}} &= \sqrt{10^{A_{\text{pass}}/10}-1}\\
\varepsilon_{\text{stop}} &= \sqrt{10^{A_{\text{stop}}/10}-1}
\end{aligned}}
\quad\Leftrightarrow\quad
\boxed{\begin{aligned}
A_{\text{pass}} &= 10\log_{10}(1+\varepsilon_{\text{pass}}^2)\\
A_{\text{stop}} &= 10\log_{10}(1+\varepsilon_{\text{stop}}^2)
\end{aligned}}
$$

The specifications of the equivalent analog filter are $\{\Omega_{\text{pass}}, \Omega_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$, or, $\{\Omega_{\text{pass}}, \Omega_{\text{stop}}, \varepsilon_{\text{pass}}, \varepsilon_{\text{stop}}\}$, where the analog frequencies are obtained by prewarping the digital frequencies:

$$\boxed{\Omega_{\text{pass}} = \tan\left(\frac{\omega_{\text{pass}}}{2}\right), \quad \Omega_{\text{stop}} = \tan\left(\frac{\omega_{\text{stop}}}{2}\right)}$$

where

$$\boxed{\omega_{\text{pass}} = \frac{2\pi f_{\text{pass}}}{f_s}, \quad \omega_{\text{stop}} = \frac{2\pi f_{\text{stop}}}{f_s}}$$

The parameters $\{\varepsilon_{\text{pass}}, \varepsilon_{\text{stop}}\}$ are useful in the design of Butterworth, Chebyshev, and elliptic filters.

To complete the design for $H(z)$, we need to first design the equivalent analog lowpass filter $H_a(s)$, and substitute into it the lowpass bilinear transformation,

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \quad \text{and} \quad H(\omega) = H_a(\Omega)\Big|_{\Omega=\tan\left(\frac{\omega}{2}\right)}$$

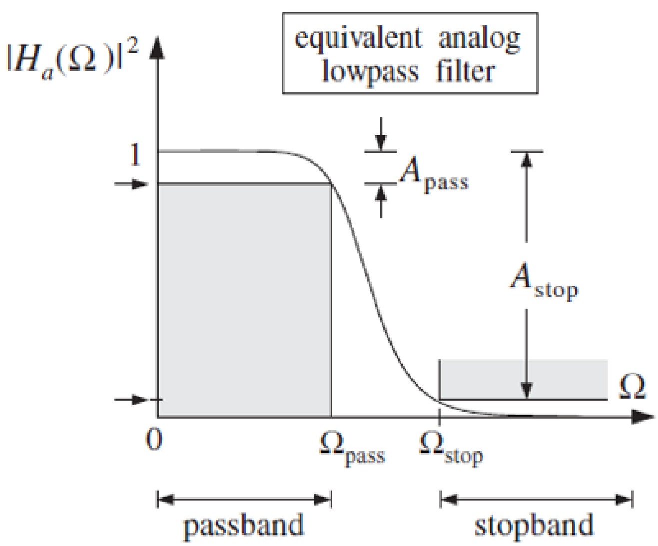
Analog Lowpass Butterworth Filters

Analog lowpass Butterworth filters are characterized by just *two* parameters: the filter order N and the 3-dB normalization frequency Ω_0. Their magnitude response is simply:

$$|H(\Omega)|^2 = \frac{1}{1 + \left(\dfrac{\Omega}{\Omega_0}\right)^{2N}}$$

and the corresponding attenuation in decibels:

$$A(\Omega) = -10\log_{10}|H(\Omega)|^2 = 10\log_{10}\left[1 + \left(\frac{\Omega}{\Omega_0}\right)^{2N}\right]$$



Note that, as N increases for fixed Ω_0, the filter becomes a better lowpass filter. At $\Omega = \Omega_0$, the magnitude response is $|H(\Omega_0)|^2 = 1/2$, or, 3-dB attenuation $A(\Omega_0) = 3$ dB. The two filter parameters $\{N, \Omega_0\}$ can be determined from the given specifications $\{\Omega_{\text{pass}}, \Omega_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$ by requiring the conditions:

$$A(\Omega_{\text{pass}}) = 10\log_{10}\left[1 + \left(\frac{\Omega_{\text{pass}}}{\Omega_0}\right)^{2N}\right] = A_{\text{pass}} = 10\log_{10}(1 + \varepsilon_{\text{pass}}^2)$$

$$A(\Omega_{\text{stop}}) = 10\log_{10}\left[1 + \left(\frac{\Omega_{\text{stop}}}{\Omega_0}\right)^{2N}\right] = A_{\text{stop}} = 10\log_{10}(1 + \varepsilon_{\text{stop}}^2)$$

To solve them for N and Ω_0, we rewrite them in the form:

$$\boxed{\begin{aligned} \left(\frac{\Omega_{\text{pass}}}{\Omega_0}\right)^{2N} &= 10^{A_{\text{pass}}/10} - 1 = \varepsilon_{\text{pass}}^2 \\ \left(\frac{\Omega_{\text{stop}}}{\Omega_0}\right)^{2N} &= 10^{A_{\text{stop}}/10} - 1 = \varepsilon_{\text{stop}}^2 \end{aligned}}$$

design equations

Taking square roots and dividing, we get an equation for N :

$$\left(\frac{\Omega_{\text{stop}}}{\Omega_{\text{pass}}}\right)^{N} = \frac{\varepsilon_{\text{stop}}}{\varepsilon_{\text{pass}}} = \sqrt{\frac{10^{A_{\text{stop}}/10} - 1}{10^{A_{\text{pass}}/10} - 1}}$$

with exact solution:

$$N_{\text{exact}} = \frac{\ln(\varepsilon_{\text{stop}}/\varepsilon_{\text{pass}})}{\ln(\Omega_{\text{stop}}/\Omega_{\text{pass}})} = \frac{\ln(e)}{\ln(w)}$$

where we defined the stopband to passband ratios:

$$e = \frac{\varepsilon_{\text{stop}}}{\varepsilon_{\text{pass}}} = \sqrt{\frac{10^{A_{\text{stop}}/10} - 1}{10^{A_{\text{pass}}/10} - 1}}\,, \quad w = \frac{\Omega_{\text{stop}}}{\Omega_{\text{pass}}}$$

Since N must be an integer, we choose it to be the *next* integer above N_{exact}, that is,

$$N = \lceil N_{\text{exact}} \rceil$$

Because N is slightly increased from its exact value, the resulting filter will be slightly better than required. But, because N is different from N_{exact}, we can no longer satisfy simultaneously both of the exact equations. So we choose to satisfy the first one exactly. This determines Ω_0 as follows:

$$\Omega_0 = \frac{\Omega_{\text{pass}}}{\left(10^{A_{\text{pass}}/10} - 1\right)^{1/2N}} = \frac{\Omega_{\text{pass}}}{\varepsilon_{\text{pass}}^{1/N}}$$

With these values of N and Ω_0, the stopband specification is more than satisfied, that is, the actual stopband attenuation will be now $A(\Omega_{\text{stop}}) > A_{\text{stop}}$. In summary, given $\{\Omega_{\text{pass}}, \Omega_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$, we solve for the parameters, N and Ω_0. We note also that we may rewrite $|H(\Omega)|^2$ in terms of the passband parameters,

$$|H(\Omega)|^2 = \frac{1}{1 + \left(\dfrac{\Omega}{\Omega_0}\right)^{2N}} = \frac{1}{1 + \varepsilon_{\text{pass}}^2 \left(\dfrac{\Omega}{\Omega_{\text{pass}}}\right)^{2N}}$$

An alternative design can be obtained by matching the stopband specification exactly, resulting in a slightly better passband, that is, $A(\Omega_{\text{stop}}) = A_{\text{stop}}$ and $A(\Omega_{\text{pass}}) < A_{\text{pass}}$. The 3-dB frequency Ω_0 is now computed by,

$$\Omega_0 = \frac{\Omega_{\text{stop}}}{\left(10^{A_{\text{stop}}/10} - 1\right)^{1/2N}} = \frac{\Omega_{\text{stop}}}{\varepsilon_{\text{stop}}^{1/N}}$$

In this case, $|H(\Omega)|^2$ can be written in terms of the stopband parameters:

$$|H(\Omega)|^2 = \frac{1}{1 + \varepsilon_{\text{stop}}^2 \left(\dfrac{\Omega}{\Omega_{\text{stop}}}\right)^{2N}} = \frac{\left(\dfrac{\Omega_{\text{stop}}}{\Omega}\right)^{2N}}{\left(\dfrac{\Omega_{\text{stop}}}{\Omega}\right)^{2N} + \varepsilon_{\text{stop}}^2}$$

The analog Butterworth transfer function $H(s)$ can be constructed from the knowledge of $\{N, \Omega_0\}$ by the method of spectral factorization, as described below. Using $s = j\Omega$ and noting that $\big(H(\Omega)\big)^* = H^*(-\Omega)$, we may write the magnitude response in terms of the variable s†

$$H(s)H^*(-s) = \frac{1}{1 + \left(\dfrac{s}{j\Omega_0}\right)^{2N}} = \frac{1}{1 + (-1)^N \left(\dfrac{s}{\Omega_0}\right)^{2N}}$$

Setting $H(s) = \dfrac{1}{D(s)}$, we have

$$\boxed{D(s)D^*(-s) = 1 + (-1)^N \left(\frac{s}{\Omega_0}\right)^{2N}}$$

Because the right-hand side is a polynomial of degree $2N$ in s, $D(s)$ will be a polynomial of degree N. There exist 2^N *different* polynomials $D(s)$ of degree N satisfying this equation.

†The notation $H^*(-s)$ denotes complex conjugation of the filter coefficients and replacement of s by $-s$, for example, $H^*(-s) = \sum a_n^*(-s)^n$ if $H(s) = \sum a_n s^n$.

But, among them, there is a unique one that has *all* its zeros in the *left-hand* s-plane. This is the one we want, because then, $H(s) = 1/D(s)$, will be stable and causal. To find this $D(s)$, we first determine all the $2N$ roots of of the above equation and then choose those that lie in the left-hand s-plane. The $2N$ solutions of,

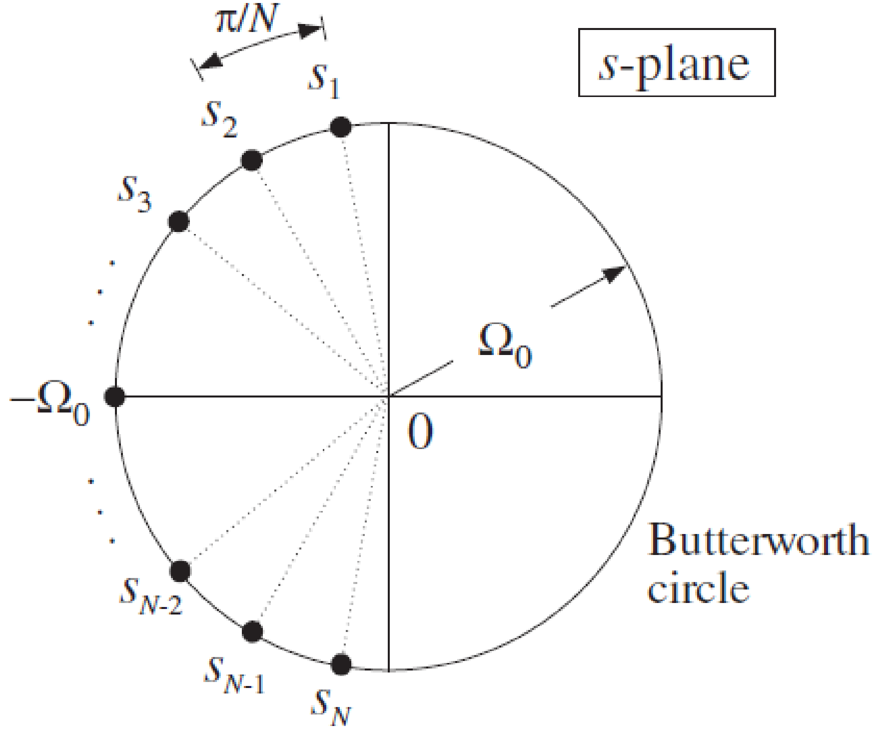
$$D(s)D^*(-s) = 1 + (-1)^N \left(\frac{s}{\Omega_0}\right)^{2N} = 0 \quad \Rightarrow \quad s^{2N} = (-1)^{N-1}{\Omega_0}^{2N}$$

are given by,

$$\boxed{s_i = \Omega_0 e^{j\theta_i}, \quad \theta_i = \frac{\pi}{2N}(N - 1 + 2i)}, \quad i = 1, 2, \ldots, N, \ldots, 2N$$

$$\boxed{s_i = \Omega_0 e^{j\theta_i}, \quad \theta_i = \frac{\pi}{2N}(N - 1 + 2i)}, \quad i = 1, 2, \ldots, N, \ldots, 2N$$

The index i is chosen such that the first N of the s_i lie in the left-hand s-plane, that is, $\pi/2 < \theta_i < 3\pi/2$ for $i = 1, 2, \ldots, N$. Because $|s_i| = \Omega_0$, all of the zeros lie on a circle of radius Ω_0, called the *Butterworth circle* and shown below.



It is seen from the figure that the s_i can be paired in complex conjugate pairs; that is, $s_N = s_1^*$, $s_{N-1} = s_2^*$, and so on.

If N is *even*, say $N = 2K$, then there are exactly K conjugate pairs, namely, $\{s_i, s_i^*\}$, $i = 1, 2, \dots, K$. In this case, $D(s)$ will factor into second-order sections as follows:

$$D(s) = D_1(s)D_2(s)\cdots D_K(s)$$

where

$$D_i(s) = \left(1 - \frac{s}{s_i}\right)\left(1 - \frac{s}{s_i^*}\right), \qquad i = 1, 2, \dots, K$$

On the other hand, if N is *odd*, say $N = 2K + 1$, there will be K conjugate pairs and one additional zero that cannot be paired and must necessarily be real-valued. That zero must lie in the left-hand s-plane and on the Butterworth circle; thus, it must be the point $s = -\Omega_0$. The polynomial $D(s)$ factors now as:

$$D(s) = D_0(s)D_1(s)D_2(s)\cdots D_K(s), \quad \text{where} \quad D_0(s) = \left(1 + \frac{s}{\Omega_0}\right)$$

The remaining factors $D_i(s)$ are the same as in both cases. They can be rewritten as factors with *real* coefficients as follows. Inserting $s_i = \Omega_0 e^{j\theta_i}$, we have for $i = 1, 2, \ldots, K$:

$$D_i(s) = \left(1 - \frac{s}{s_i}\right)\left(1 - \frac{s}{s_i^*}\right) = 1 - 2\frac{s}{\Omega_0}\cos\theta_i + \frac{s^2}{\Omega_0^2}$$

Inserting these factors into the Butterworth analog transfer function $H(s) = 1/D(s)$, we can express it as a cascade of second-order sections:

$$H(s) = H_0(s)H_1(s)H_2(s)\cdots H_K(s)$$

N-th order analog Butterworth filter

where,

$$H_0(s) = \begin{cases} 1, & \text{if } N = 2K \\ \dfrac{1}{1 + \dfrac{s}{\Omega_0}}, & \text{if } N = 2K+1 \end{cases}$$

$$H_i(s) = \frac{1}{1 - 2\dfrac{s}{\Omega_0}\cos\theta_i + \dfrac{s^2}{\Omega_0^2}}, \qquad i = 1, 2, \ldots, K$$

$$\theta_i = \frac{\pi}{2N}(N - 1 + 2i)$$

Example:
The Butterworth polynomials $D(s)$ of orders 1–7 and unity 3-dB normalization frequency $\Omega_0 = 1$ are shown in the table below. For other values of Ω_0, s must be replaced by s/Ω_0 in each table entry.

The coefficients of s of the second-order sections are the cosine factors, $-2\cos\theta_i$. For example, in the case $N = 7$, we have $K = 3$ and the three θ's are calculated from:

$$\theta_i = \frac{\pi}{14}(6+2i) = \frac{8\pi}{14},\ \frac{10\pi}{14},\ \frac{12\pi}{14}, \quad \text{for} \quad i = 1, 2, 3$$

$$-2\cos\theta_i = 0.4450,\ 1.2470,\ 1.8019$$

The corresponding Butterworth filters $H(s)$ of orders 1–7 are obtained as the inverses of the table entries.

Butterworth polynomials

N	K	$\theta_1, \theta_2, \ldots, \theta_K$	$D(s)$
1	0		$(1+s)$
2	1	$\frac{3\pi}{4}$	$(1+1.4142s+s^2)$
3	1	$\frac{4\pi}{6}$	$(1+s)(1+s+s^2)$
4	2	$\frac{5\pi}{8}, \frac{7\pi}{8}$	$(1+0.7654s+s^2)(1+1.8478s+s^2)$
5	2	$\frac{6\pi}{10}, \frac{8\pi}{10}$	$(1+s)(1+0.6180s+s^2)(1+1.6180s+s^2)$
6	3	$\frac{7\pi}{12}, \frac{9\pi}{12}, \frac{11\pi}{12}$	$(1+0.5176s+s^2)(1+1.4142s+s^2)(1+1.9319s+s^2)$
7	3	$\frac{8\pi}{14}, \frac{10\pi}{14}, \frac{12\pi}{14}$	$(1+s)(1+0.4450s+s^2)(1+1.2470s+s^2)(1+1.8019s+s^2)$

Example:
Determine the 2^N possible Nth degree Butterworth polynomials $D(s)$, for the cases $N = 2$ and $N = 3$. Take $\Omega_0 = 1$.

Solution:
For $N = 2$, we must find all second-degree polynomials that satisfy, $D(s)D^*(-s) = 1 + (-1)^2 s^4$. They are:

$$\begin{array}{lcl} D(s) = 1 + \sqrt{2}s + s^2 & & D^*(-s) = 1 - \sqrt{2}s + s^2 \\ D(s) = 1 - \sqrt{2}s + s^2 & \Rightarrow & D^*(-s) = 1 + \sqrt{2}s + s^2 \\ D(s) = 1 + js^2 & & D^*(-s) = 1 - js^2 \\ D(s) = 1 - js^2 & & D^*(-s) = 1 + js^2 \end{array}$$

Only the first one has all of its zeros in the left-hand s-plane.

Similarly, for $N = 3$, the $2^N = 2^3 = 8$ different third-degree polynomials $D(s)$ are:

$$
\begin{array}{lcl}
D(s) = (1+s)(1+s+s^2) & & D^*(-s) = (1-s)(1-s+s^2) \\
D(s) = (1+s)(1-s+s^2) & & D^*(-s) = (1-s)(1+s+s^2) \\
D(s) = (1+s)(1-s^2e^{2j\pi/3}) & & D^*(-s) = (1-s)(1-s^2e^{-2j\pi/3}) \\
D(s) = (1+s)(1-s^2e^{-2j\pi/3}) & \Rightarrow & D^*(-s) = (1-s)(1-s^2e^{2j\pi/3}) \\
D(s) = (1-s)(1-s^2e^{-2j\pi/3}) & & D^*(-s) = (1+s)(1-s^2e^{2j\pi/3}) \\
D(s) = (1-s)(1-s^2e^{2j\pi/3}) & & D^*(-s) = (1+s)(1-s^2e^{-2j\pi/3}) \\
D(s) = (1-s)(1+s+s^2) & & D^*(-s) = (1+s)(1-s+s^2) \\
D(s) = (1-s)(1-s+s^2) & & D^*(-s) = (1+s)(1+s+s^2)
\end{array}
$$

They all satisfy, $D(s)D^*(-s) = 1 + (-1)^3 s^6$, but, only the first one has its zeros in the left-hand s-plane.

Digital Lowpass Filters

Under the lowpass bilinear transformation, the lowpass analog filter will b transformed into a lowpass digital filter. Each analog 2nd-order section wi be transformed into a 2nd-order section of the digital filter, as follows:

$$H_i(z) = \left. \frac{1}{1 - 2\dfrac{s}{\Omega_0}\cos\theta_i + \dfrac{s^2}{\Omega_0^2}} \right|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{G_i(1+z^{-1})^2}{1 + a_{i1}z^{-1} + a_{i2}z^{-2}}$$

If N is odd, then there is also a first-order section:

$$H_0(z) = \left. \frac{1}{1 + \dfrac{s}{\Omega_0}} \right|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{G_0(1+z^{-1})}{1 + a_{01}z^{-1}}$$

The overall transfer function of the designed lowpass digital filter will be:

$$H(z) = H_0(z)H_1(z)H_2(z)\cdots H_K(z)$$

If N is even, we may set $H_0(z) = 1$.

where the filter coefficients G_i, a_{i1}, a_{i2} are easily found to be:

$$G_i = \frac{\Omega_0^2}{1 - 2\Omega_0 \cos\theta_i + {\Omega_0}^2}$$

$$a_{i1} = \frac{2(\Omega_0^2 - 1)}{1 - 2\Omega_0 \cos\theta_i + {\Omega_0}^2}$$

$$a_{i2} = \frac{1 + 2\Omega_0 \cos\theta_i + \Omega_0^2}{1 - 2\Omega_0 \cos\theta_i + {\Omega_0}^2}$$

for, $i = 1, 2, \ldots, K$, and for the first-order section,

$$G_0 = \frac{\Omega_0}{\Omega_0 + 1}, \qquad a_{01} = \frac{\Omega_0 - 1}{\Omega_0 + 1}$$

The 3-dB frequency f_0 in Hz of the digital filter is related to the analog Butterworth parameter Ω_0 by,

$$\Omega_0 = \tan\left(\frac{\omega_0}{2}\right) = \tan\left(\frac{\pi f_0}{f_s}\right) \quad \Rightarrow \quad \boxed{f_0 = \frac{f_s}{\pi}\arctan(\Omega_0)}$$

The filter sections have zeros at $z = -1$, that is, at the Nyquist frequency, $\omega = \pi$. Setting $\Omega = \tan(\omega/2)$, the magnitude response of the designed digital filter can be expressed simply as follows:

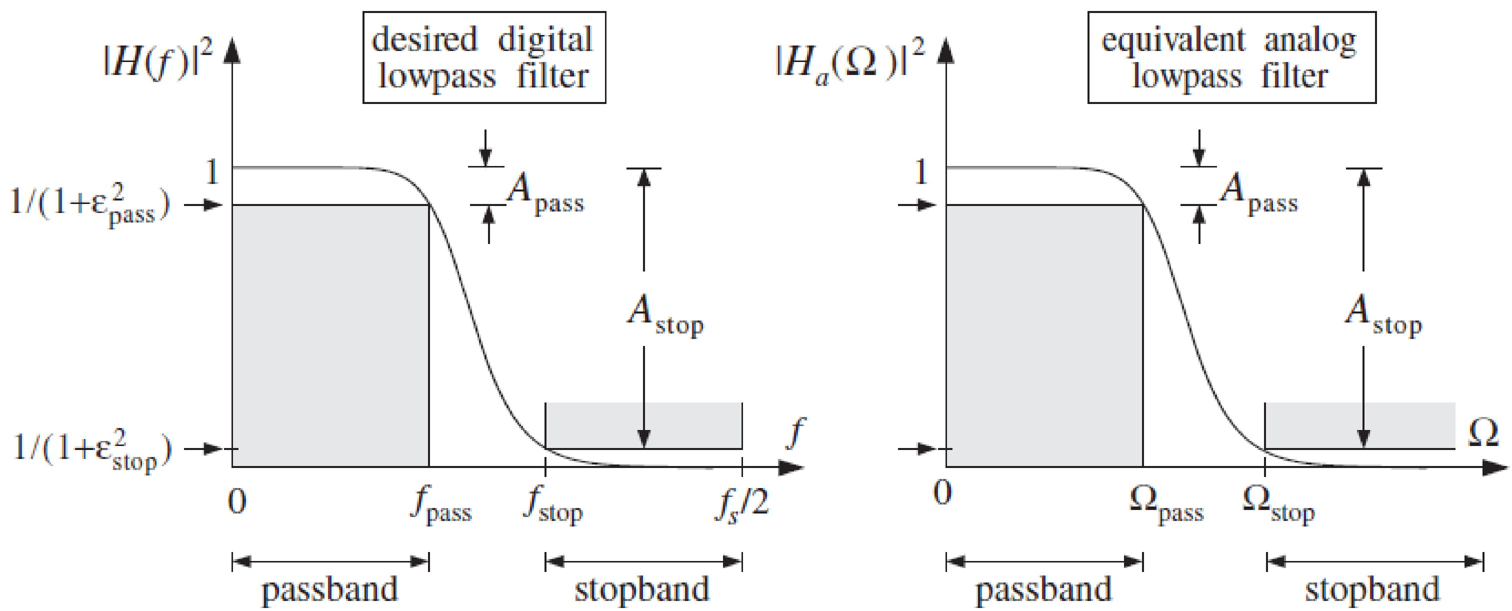
$$|H(\omega)|^2 = |H_a(\Omega)|^2 = \frac{1}{1 + \left(\Omega/\Omega_0\right)^{2N}} = \frac{1}{1 + \left(\tan(\omega/2)/\Omega_0\right)^{2N}}$$

Note also that each second-order section has unity gain at zero frequency, $f = 0$, $\omega = 0$, or $z = 1$. Indeed, setting $z = 1$, we obtain the following conditions, which can be verified from the above definitions,

$$\frac{4G_i}{1 + a_{i1} + a_{i2}} = 1 \qquad \text{and} \qquad \frac{2G_0}{1 + a_{01}} = 1$$

In summary, the design steps for a lowpass digital filter with given specifications $\{f_{\text{pass}}, f_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$ are:

1. Calculate the digital frequencies $\{\omega_{\text{pass}}, \omega_{\text{stop}}\}$ and the corresponding prewarped versions $\{\Omega_{\text{pass}}, \Omega_{\text{stop}}\}$.
2. Calculate the order N and 3-dB frequency Ω_0 of the equivalent lowpass analog Butterworth filter based on the transformed specifications $\{\Omega_{\text{pass}}, \Omega_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$.
3. The transfer function of the desired lowpass digital filter is then obtained in the above cascaded form of first- and second-order sections.



Example:

Using the bilinear transformation and a lowpass analog Butterworth prototype, design a lowpass digital filter operating at a rate of 20 kHz and having passband extending to 4 kHz with maximum passband attenuation of 0.5 dB, and stopband starting at 5 kHz with a minimum stopband attenuation of 10 dB. Then, redesign it such that its magnitude response satisfies

$$1 \geq |H(f)|^2 \geq 0.98$$

in its passband, and

$$|H(f)|^2 \leq 0.02$$

in its stopband.

Solution:

The digital frequencies in radians per sample are:

$$\omega_{\text{pass}} = \frac{2\pi f_{\text{pass}}}{f_s} = \frac{2\pi \cdot 4}{20} = 0.4\pi, \quad \omega_{\text{stop}} = \frac{2\pi f_{\text{stop}}}{f_s} = \frac{2\pi \cdot 5}{20} = 0.5\pi$$

and their prewarped versions:

$$\Omega_{\text{pass}} = \tan\left(\frac{\omega_{\text{pass}}}{2}\right) = 0.7265, \quad \Omega_{\text{stop}} = \tan\left(\frac{\omega_{\text{stop}}}{2}\right) = 1$$

With the given values, $A_{\text{pass}} = 0.5$ dB and $A_{\text{stop}} = 10$ dB, we calculate the parameters $\{\varepsilon_{\text{pass}}, \varepsilon_{\text{stop}}\}$:

$$\varepsilon_{\text{pass}} = \sqrt{10^{A_{\text{pass}}/10} - 1} = \sqrt{10^{0.5/10} - 1} = 0.3493$$

$$\varepsilon_{\text{stop}} = \sqrt{10^{A_{\text{stop}}/10} - 1} = \sqrt{10^{10/10} - 1} = 3$$

Then, the exact N and its rounded-up value are,

$$N_{\text{exact}} = \frac{\ln(e)}{\ln(w)} = \frac{\ln(\varepsilon_{\text{stop}}/\varepsilon_{\text{pass}})}{\ln(\Omega_{\text{stop}}/\Omega_{\text{pass}})} = \frac{\ln(3/0.3493)}{\ln(1/0.7265)} = 6.73 \quad \Rightarrow \quad N = 7$$

Thus, there is one first-order section $H_0(z)$ and three second-order sections. Next we calculate Ω_0 and its corresponding value in Hz:

$$\Omega_0 = \frac{\Omega_{\text{pass}}}{\varepsilon_{\text{pass}}^{1/N}} = \frac{0.7265}{(0.3493)^{1/7}} = 0.8443$$

$$f_0 = \frac{f_s}{\pi} \arctan(\Omega_0) = \frac{20}{\pi} \arctan(0.8443) = 4.4640 \text{ kHz}$$

The Butterworth angles θ_1, θ_2, θ_3 are calculated from,

$$\theta_i = \frac{\pi}{2N}(N - 1 + 2i) = \frac{\pi}{14}(6 + 2i)\,, \quad i = 1, 2, 3$$

and the calculated SOS coefficients are then found to be:

i	G_i	a_{i1}	a_{i2}
0	0.4578	−0.0844	
1	0.3413	−0.2749	0.6402
2	0.2578	−0.2076	0.2386
3	0.2204	−0.1775	0.0592

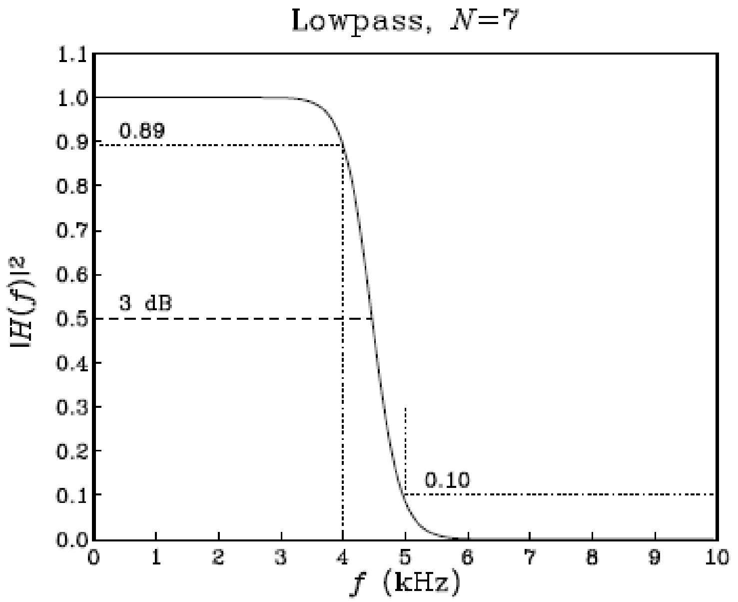
resulting in the transfer function:

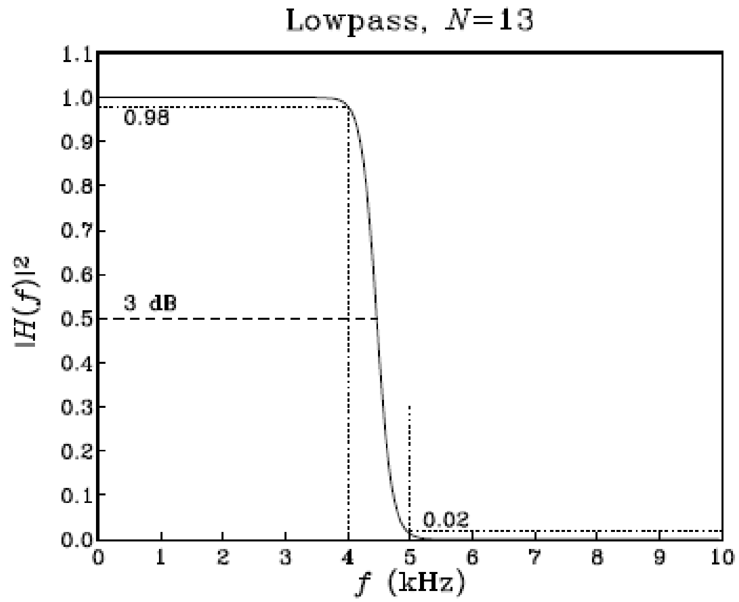
$$\begin{aligned} H(z) &= H_0(z)H_1(z)H_2(z)H_3(z) \\ &= \frac{0.4578(1 + z^{-1})}{1 - 0.0844z^{-1}} \cdot \frac{0.3413(1 + z^{-1})^2}{1 - 0.2749z^{-1} + 0.6402z^{-2}} \\ &\quad \cdot \frac{0.2578(1 + z^{-1})^2}{1 - 0.2076z^{-1} + 0.2386z^{-2}} \cdot \frac{0.2204(1 + z^{-1})^2}{1 - 0.1775z^{-1} + 0.0592z^{-2}} \end{aligned}$$

The left graph of the figure below shows the magnitude response squared, $|H(f)|^2$. The brick-wall specifications and the 3-dB line intersecting the response at $f = f_0$ are shown on the graph.

$$|H(f)|^2 = \frac{1}{1 + \left(\tan(\pi f/f_s)/\Omega_0\right)^{2N}} = \frac{1}{1 + \left(\tan(\pi f/20)/0.8443\right)^{14}}$$

The passband attenuation in absolute units is $10^{-0.5/10} = 0.89125$ and the stopband attenuation $10^{-10/10} = 0.1$. Note that the actual stopband attenuation at $f = f_{\text{stop}} = 5$ kHz is slightly better than required, that is, $A(f_{\text{stop}}) = 10.68$ dB.





The second filter has more stringent specifications. The desired passband attenuation is $A_{\text{pass}} = -10\log_{10}(0.98) = 0.0877$ dB, and the stopband attenuation $A_{\text{stop}} = -10\log_{10}(0.02) = 16.9897$ dB. With these values, we find the design parameters $\{\varepsilon_{\text{pass}}, \varepsilon_{\text{stop}}\} = \{0.1429, 7\}$ and:

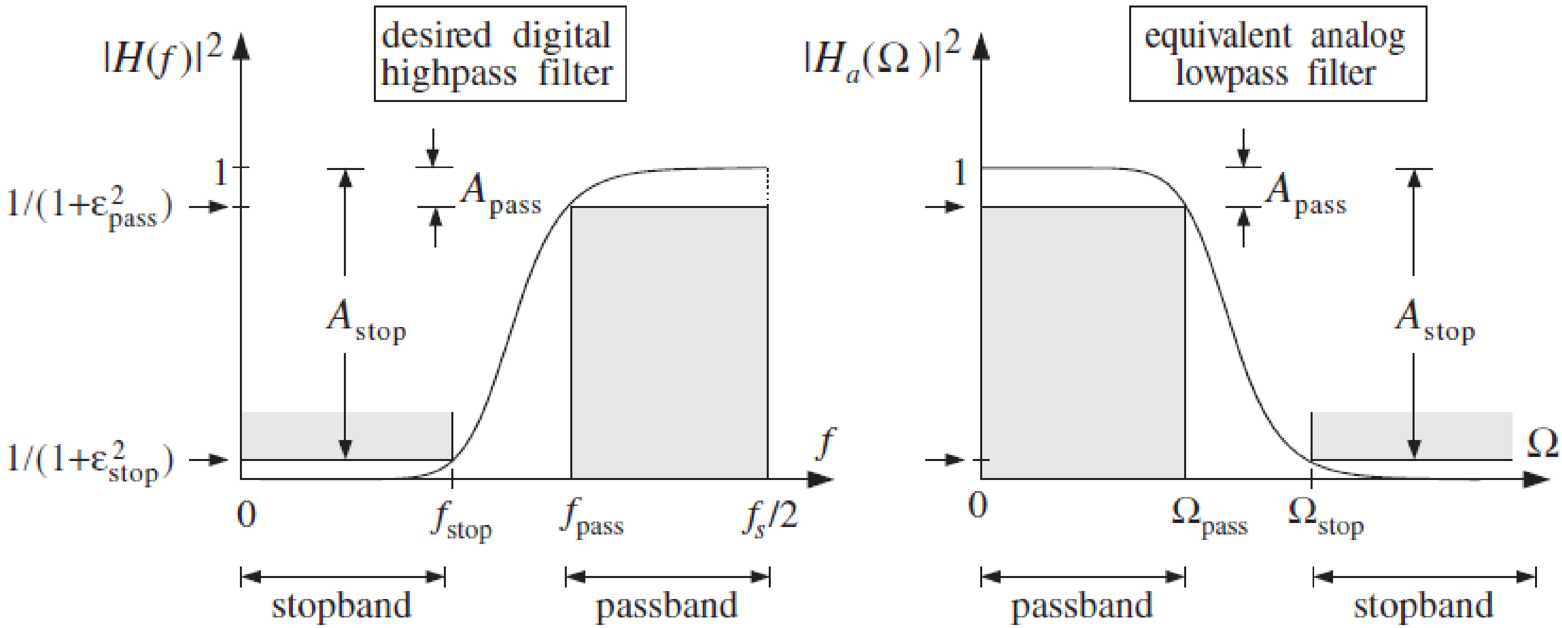
$$N_{\text{exact}} = 12.18, \quad N = 13, \quad \Omega_0 = 0.8439, \quad f_0 = 4.4622\text{ kHz}$$

The digital filter will have one first-order and six second-order sections. The SOS coefficients were calculated with the MATLAB function **lhbutt.m** of I2SP Appendix.

i	G_i	a_{i1}	a_{i2}
0	0.4577	−0.0847	
1	0.3717	−0.3006	0.7876
2	0.3082	−0.2492	0.4820
3	0.2666	−0.2156	0.2821
4	0.2393	−0.1935	0.1508
5	0.2221	−0.1796	0.0679
6	0.2125	−0.1718	0.0219

Its magnitude response is shown in the right graph of the above figure. As is always the case, making the specifications more stringent results in higher order N.

Digital Highpass Filters



The mapping is now carried out by the highpass version of the bilinear transformation,

$$s = \frac{1+z^{-1}}{1-z^{-1}}, \qquad \Omega = -\cot\left(\frac{\omega}{2}\right), \qquad \omega = \frac{2\pi f}{f_s}$$

It maps the point $z = -1$ to $s = 0$, or equivalently, the center of the passband of the highpass filter at $\omega = \pi$ to the center of the passband of the lowpass filter at $\Omega = 0$.

The prewarped versions of the passband and stopband frequencies are computed as follows:

$$\Omega_{\text{pass}} = \cot\left(\frac{\omega_{\text{pass}}}{2}\right) = \cot\left(\frac{\pi f_{\text{pass}}}{f_s}\right)$$
$$\Omega_{\text{stop}} = \cot\left(\frac{\omega_{\text{stop}}}{2}\right) = \cot\left(\frac{\pi f_{\text{stop}}}{f_s}\right)$$

We should have used $\Omega_{\text{pass}} = -\cot(\omega_{\text{pass}}/2)$. However, as far as the determination of the parameters N and Ω_0 is concerned, it does not matter whether we use positive or negative signs because we are working only with the magnitude response of the analog filter, which is even as a function of Ω. The analog lowpass Butterworth parameters N, Ω_0 are now determined from the values of $\{\Omega_{\text{pass}}, \Omega_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$.

Under the highpass bilinear transformation, each SOS of the analog filter will be transformed into an SOS of the digital filter, as follows,

$$H_i(z) = \left. \frac{1}{1 - 2\dfrac{s}{\Omega_0}\cos\theta_i + \dfrac{s^2}{\Omega_0^2}} \right|_{s=\frac{1+z^{-1}}{1-z^{-1}}} = \frac{G_i(1-z^{-1})^2}{1 + a_{i1}z^{-1} + a_{i2}z^{-2}}$$

If N is odd, then there is also a first-order section given by

$$H_0(z) = \left. \frac{1}{1 + \dfrac{s}{\Omega_0}} \right|_{s=\frac{1+z^{-1}}{1-z^{-1}}} = \frac{G_0(1-z^{-1})}{1 + a_{01}z^{-1}}$$

The overall transfer function of the highpass digital filter will be,

$$H(z) = H_0(z)H_1(z)H_2(z)\cdots H_K(z)$$

If N is even, we may set $H_0(z) = 1$.

where the filter coefficients G_i, a_{i1}, a_{i2} are easily found to be

$$G_i = \frac{\Omega_0^2}{1 - 2\Omega_0 \cos\theta_i + {\Omega_0}^2}$$

$$a_{i1} = -\frac{2(\Omega_0^2 - 1)}{1 - 2\Omega_0 \cos\theta_i + {\Omega_0}^2}$$

$$a_{i2} = \frac{1 + 2\Omega_0 \cos\theta_i + \Omega_0^2}{1 - 2\Omega_0 \cos\theta_i + {\Omega_0}^2}$$

for, $i = 1, 2, \ldots, K$, and for the first-order section,

$$G_0 = \frac{\Omega_0}{\Omega_0 + 1}, \qquad a_{01} = -\frac{\Omega_0 - 1}{\Omega_0 + 1}$$

The 3-dB frequency f_0 of the designed filter may be calculated from:

$$\Omega_0 = \cot\left(\frac{\omega_0}{2}\right) = \cot\left(\frac{\pi f_0}{f_s}\right) \quad \Rightarrow \quad \boxed{f_0 = \frac{f_s}{\pi}\arctan\left(\frac{1}{\Omega_0}\right)}$$

and the magnitude response from:

$$|H(\omega)|^2 = \frac{1}{1 + \big(\cot(\omega/2)/\Omega_0\big)^{2N}}$$

Note the similarities and differences between the highpass and lowpass cases: The coefficients G_i and a_{i2} are the same, but a_{i1} has reverse sign. Also, the numerator of the SOS is now $(1 - z^{-1})^2$ instead of $(1 + z^{-1})^2$, resulting in a zero at $z = 1$ or $\omega = 0$. These changes are easily understood by noting that the lowpass bilinear transformation becomes the highpass one given by under the substitution $z \to -z$.

Example: Using the bilinear transformation and a lowpass analog Butterworth prototype, design a highpass digital filter operating at a rate of 20 kHz and having passband starting at 5 kHz with maximum passband attenuation of 0.5 dB, and stopband ending at 4 kHz with a minimum stopband attenuation of 10 dB. Then, redesign it such that its magnitude response satisfies

$$1 \geq |H(f)|^2 \geq 0.98$$

in the passband, and

$$|H(f)|^2 \leq 0.02$$

in the stopband.

Solution:
The digital frequencies and their prewarped versions are:

$$\begin{aligned} \omega_{\text{pass}} &= \frac{2\pi f_{\text{pass}}}{f_s} = \frac{2\pi \cdot 5}{20} = 0.5\pi, \\ \omega_{\text{stop}} &= \frac{2\pi f_{\text{stop}}}{f_s} = \frac{2\pi \cdot 4}{20} = 0.4\pi, \end{aligned} \quad \Rightarrow \quad \begin{aligned} \Omega_{\text{pass}} &= \cot\left(\frac{\omega_{\text{pass}}}{2}\right) = 1 \\ \Omega_{\text{stop}} &= \cot\left(\frac{\omega_{\text{stop}}}{2}\right) = 1.3764 \end{aligned}$$

The dB attenuations $\{A_{\text{pass}}, A_{\text{stop}}\} = \{0.5, 10\}$ correspond to $\{\varepsilon_{\text{pass}}, \varepsilon_{\text{stop}}\} = \{0.3493, 3\}$. Then, we solve for N,

$$N_{\text{exact}} = \frac{\ln(\varepsilon_{\text{stop}}/\varepsilon_{\text{pass}})}{\ln(\Omega_{\text{stop}}/\Omega_{\text{pass}})} = \frac{\ln(3/0.3493)}{\ln(1.3764/1)} = 6.73 \quad \Rightarrow \quad N = 7$$

Thus, there is one first-order section $H_0(z)$ and three second-order sections. The calculated parameter Ω_0 is now,

$$\Omega_0 = \frac{\Omega_{\text{pass}}}{\left(10^{A_{\text{pass}}/10} - 1\right)^{1/2N}} = \frac{\Omega_{\text{pass}}}{\varepsilon_{\text{pass}}^{1/N}} = \frac{1}{(0.3493)^{1/7}} = 1.1621$$

The SOS coefficients are found to be

i	G_i	a_{i1}	a_{i2}
0	0.5375	−0.0750	
1	0.4709	−0.2445	0.6393
2	0.3554	−0.1845	0.2372
3	0.3039	−0.1577	0.0577

resulting in the transfer function:

$$\begin{aligned} H(z) &= H_0(z)H_1(z)H_2(z)H_3(z) \\ &= \frac{0.5375(1-z^{-1})}{1-0.0750z^{-1}} \cdot \frac{0.4709(1-z^{-1})^2}{1-0.2445z^{-1}+0.6393z^{-2}} \\ &\quad \cdot \frac{0.3554(1-z^{-1})^2}{1-0.1845z^{-1}+0.2372z^{-2}} \cdot \frac{0.3039(1-z^{-1})^2}{1-0.1577z^{-1}+0.0577z^{-2}} \end{aligned}$$

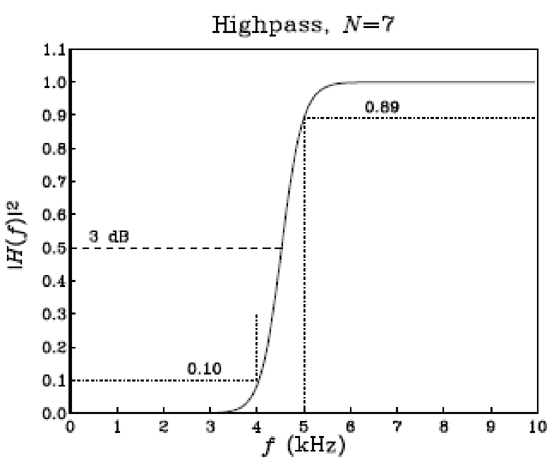
As in previous lowpass example, the second filter has passband and stopband attenuations: $A_{\text{pass}} = -10\log_{10}(0.98) = 0.0877$ dB and $A_{\text{stop}} = -10\log_{10}(0.02) = 16.9897$ dB. With these values, we find the design parameters $\{\varepsilon_{\text{pass}}, \varepsilon_{\text{stop}}\} = \{0.1429, 7\}$ and:

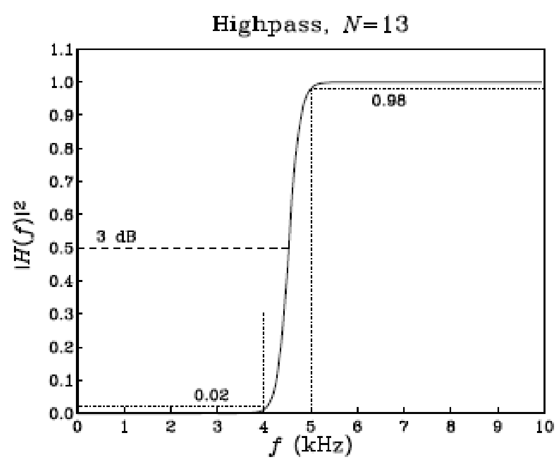
$$N_{\text{exact}} = 12.18, \quad N = 13, \quad \Omega_0 = 1.1615, \quad f_0 = 4.5253\text{ kHz}$$

The coefficients of the first- and second-order sections are:

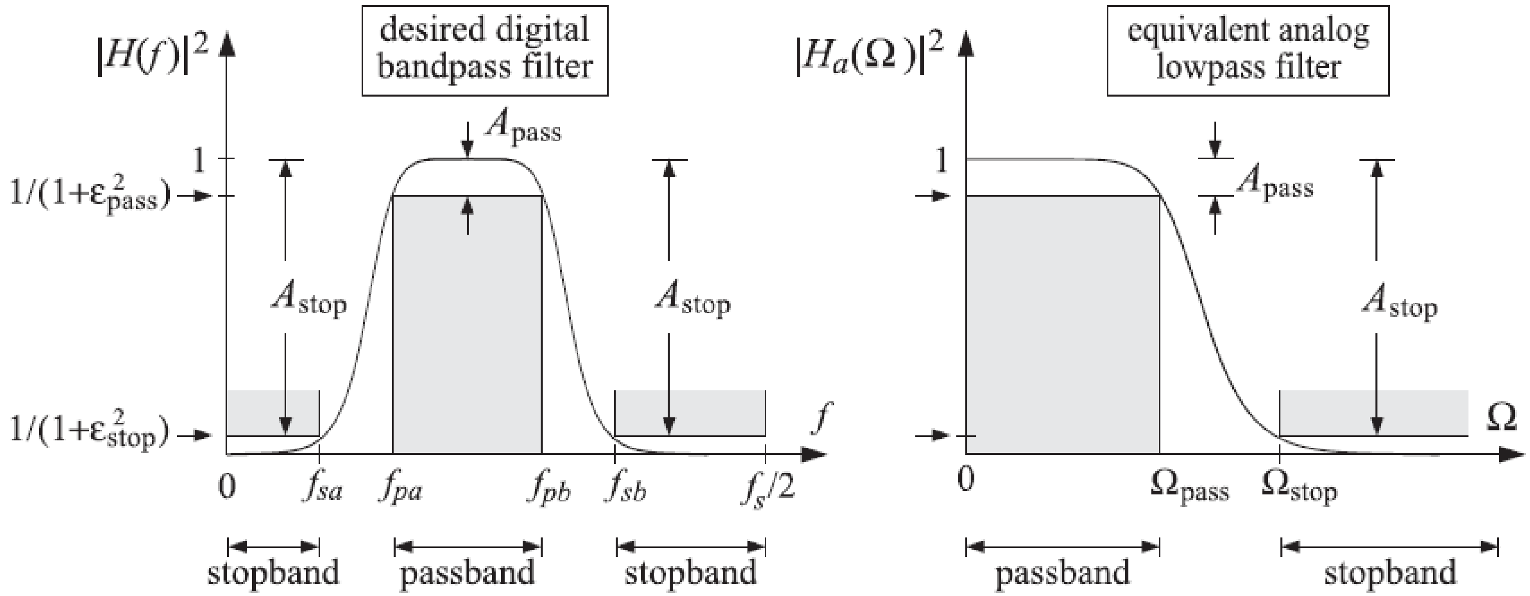
i	G_i	a_{i1}	a_{i2}
0	0.5374	−0.0747	
1	0.5131	−0.2655	0.7870
2	0.4252	−0.2200	0.4807
3	0.3677	−0.1903	0.2806
4	0.3300	−0.1708	0.1493
5	0.3062	−0.1584	0.0663
6	0.2930	−0.1516	0.0203

The above design steps are implemented in the I2SP function, **lhbutt.m**. The magnitude responses of the two designs are shown below.





Digital Bandpass Filters



The specifications are the quantities $\{f_{pa}, f_{pb}, f_{sa}, f_{sb}, A_{\text{pass}}, A_{\text{stop}}\}$, defining the frequency bands:

– the passband range $f_{pa} \le f \le f_{pb}$,

– the left stopband $0 \le f \le f_{sa}$, and

– the right stopband $f_{sb} \le f \le f_s/2$.

The stopband attenuations were assumed to be equal in the two stopbands; if not, we may design the filter based on the maximum of the two.

The bandpass version of the bilinear transformation and the corresponding frequency mapping are in this case:

$$s = \frac{1 - 2cz^{-1} + z^{-2}}{1 - z^{-2}}, \qquad \Omega = \frac{c - \cos\omega}{\sin\omega}, \qquad \omega = \frac{2\pi f}{f_s}$$

A new parameter c has been introduced. Note that $c = 1$ recovers the lowpass case, and $c = -1$ the highpass one. The parameter c is required to be $|c| \leq 1$ in order to map the left-hand s-plane into the inside of the unit circle in the z-plane.

Therefore, we may set $c = \cos\omega_c$, for some value of ω_c. The center of the analog passband $\Omega = 0$ corresponds to $\cos\omega = c = \cos\omega_c$, or, $\omega = \omega_c$. Therefore, ω_c may be thought of as the "center" frequency of the bandpass filter (although it need not be exactly at the center of the passband).

The given bandpass specifications, must be mapped onto the specifications of the equivalent analog lowpass filter, $\{\Omega_{\text{pass}}, \Omega_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$.

This can be done as follows. We require that the passband $[f_{pa}, f_{pb}]$ of the digital filter be mapped onto the entire passband $[-\Omega_{\text{pass}}, \Omega_{\text{pass}}]$ of the analog filter. This requires that:

$$-\Omega_{\text{pass}} = \frac{c - \cos\omega_{pa}}{\sin\omega_{pa}}$$

$$\Omega_{\text{pass}} = \frac{c - \cos\omega_{pb}}{\sin\omega_{pb}}$$

where $\omega_{pa} = 2\pi f_{pa}/f_s$ and $\omega_{pb} = 2\pi f_{pb}/f_s$. By adding them, we solve for c. Then, inserting the computed value of c into one or the other we find Ω_{pass}. The resulting solution is:

$$c = \frac{\sin(\omega_{pa} + \omega_{pb})}{\sin\omega_{pa} + \sin\omega_{pb}}\,, \qquad \Omega_{\text{pass}} = \left|\frac{c - \cos\omega_{pb}}{\sin\omega_{pb}}\right|$$

Note that for ω_{pa}, ω_{pb} in the interval $[0, \pi]$, the above expression for c implies $|c| \le 1$, as required for stability.

Next, we compute the two numbers:

$$\Omega_{sa} = \frac{c - \cos\omega_{sa}}{\sin\omega_{sa}}, \qquad \Omega_{sb} = \frac{c - \cos\omega_{sb}}{\sin\omega_{sb}}$$

where $\omega_{sa} = 2\pi f_{sa}/f_s$ and $\omega_{sb} = 2\pi f_{sb}/f_s$. Ideally, the stopband of the digital filter should map exactly onto the stopband of the analog filter so that we should have $\Omega_{sb} = \Omega_{\text{stop}}$ and $\Omega_{sa} = -\Omega_{\text{stop}}$. But this is impossible because c has already been determined.

Because the Butterworth magnitude response is a *monotonically decreasing* function of Ω, it is enough to choose the smallest of the two stopbands defined above. Thus, we define:

$$\Omega_{\text{stop}} = \min\big(|\Omega_{sa}|, |\Omega_{sb}|\big)$$

With the computed values of Ω_{pass} and Ω_{stop}, we proceed to compute the Butterworth parameters N and Ω_0 and the corresponding analog filter SOS sections as before.

Because s is quadratic in z, the substitution of s into these SOSs will give rise to *fourth-order* sections in z:

$$H_i(z) = \frac{1}{1 - 2\dfrac{s}{\Omega_0}\cos\theta_i + \dfrac{s^2}{\Omega_0^2}}\Bigg|_{s=\frac{1-2cz^{-1}+z^{-2}}{1-z^{-2}}}$$

$$= \frac{G_i(1 - z^{-2})^2}{1 + a_{i1}z^{-1} + a_{i2}z^{-2} + a_{i3}z^{-3} + a_{i4}z^{-4}}$$

where, for $i = 1, 2, \ldots, K$:

$$G_i = \frac{\Omega_0^2}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}$$

$$a_{i1} = \frac{4c(\Omega_0\cos\theta_i - 1)}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2} \qquad a_{i2} = \frac{2(2c^2 + 1 - \Omega_0^2)}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}$$

$$a_{i3} = -\frac{4c(\Omega_0\cos\theta_i + 1)}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2} \qquad a_{i4} = \frac{1 + 2\Omega_0\cos\theta_i + \Omega_0^2}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}$$

If N is odd, then there is also a first-order section in s which becomes a second-order section in z:

$$H_0(z) = \left. \frac{1}{1 + \dfrac{s}{\Omega_0}} \right|_{s=\frac{1-2cz^{-1}+z^{-2}}{1-z^{-2}}} = \frac{G_0(1 - z^{-2})}{1 + a_{01}z^{-1} + a_{02}z^{-2}}$$

where

$$G_0 = \frac{\Omega_0}{1 + \Omega_0}, \qquad a_{01} = -\frac{2c}{1 + \Omega_0}, \qquad a_{02} = \frac{1 - \Omega_0}{1 + \Omega_0}$$

The overall transfer function of the bandpass digital filter will be given as the cascade of fourth-order sections with the possibility of one SOS:

$$H(z) = H_0(z)H_1(z)H_2(z)\cdots H_K(z)$$

The order of the digital filter is $2N$, because s is quadratic in z. The filter sections have zeros at $z = \pm 1$, that is, $\omega = 0$ and $\omega = \pi$.

The left and right 3-dB frequencies can be calculated from the equations:

$$\frac{c - \cos\omega_0}{\sin\omega_0} = \mp \Omega_0$$

They can be solved by writing $\cos\omega_0$ and $\sin\omega_0$ in terms of $\tan(\omega_0/2)$, solving the resulting quadratic equation, and picking the positive solutions:

$$\tan\left(\frac{\omega_{0a}}{2}\right) = \tan\left(\frac{\pi f_{0a}}{f_s}\right) = \frac{\sqrt{\Omega_0^2 + 1 - c^2} - \Omega_0}{1 + c}$$

$$\tan\left(\frac{\omega_{0b}}{2}\right) = \tan\left(\frac{\pi f_{0b}}{f_s}\right) = \frac{\sqrt{\Omega_0^2 + 1 - c^2} + \Omega_0}{1 + c}$$

Example:
Using the bilinear transformation and a lowpass analog Butterworth prototype, design a bandpass digital filter operating at a rate of 20 kHz and having left and right passband frequencies of 2 and 4 kHz, and left and right stopband frequencies of 1.5 and 4.5 kHz. The maximum passband attenuation is required to be 0.5 dB, and the minimum stopband attenuation 10 dB. Then, redesign it such that its magnitude response satisfies,

$$1 \geq |H(f)|^2 \geq 0.98$$

in the passband, and

$$|H(f)|^2 \leq 0.02$$

in the stopbands.

Solution:
The digital passband frequencies are:

$$\omega_{pa} = \frac{2\pi f_{pa}}{f_s} = \frac{2\pi \cdot 2}{20} = 0.2\pi, \quad \omega_{pb} = \frac{2\pi f_{pb}}{f_s} = \frac{2\pi \cdot 4}{20} = 0.4\pi$$

Then, we calculate c and Ω_{pass}:

$$c = \frac{\sin(\omega_{pa} + \omega_{pb})}{\sin \omega_{pa} + \sin \omega_{pb}} = 0.6180, \qquad \Omega_{\text{pass}} = \left| \frac{c - \cos \omega_{pb}}{\sin \omega_{pb}} \right| = 0.3249$$

With the stopband digital frequencies:

$$\omega_{sa} = \frac{2\pi f_{sa}}{f_s} = \frac{2\pi \cdot 1.5}{20} = 0.15\pi, \quad \omega_{sb} = \frac{2\pi f_{sb}}{f_s} = \frac{2\pi \cdot 4.5}{20} = 0.45\pi$$

we calculate:

$$\Omega_{sa} = \frac{c - \cos\omega_{sa}}{\sin\omega_{sa}} = -0.6013, \quad \Omega_{sb} = \frac{c - \cos\omega_{sb}}{\sin\omega_{sb}} = 0.4674$$

and $\Omega_{\text{stop}} = \min\big(|\Omega_{sa}|, |\Omega_{sb}|\big) = 0.4674$. The analog filter with the specifications $\{\Omega_{\text{pass}}, \Omega_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$ has parameters $\{\varepsilon_{\text{pass}}, \varepsilon_{\text{stop}}\} = \{0.3493, 3\}$ and:

$$N_{\text{exact}} = 5.92, \quad N = 6, \quad \Omega_0 = 0.3872$$

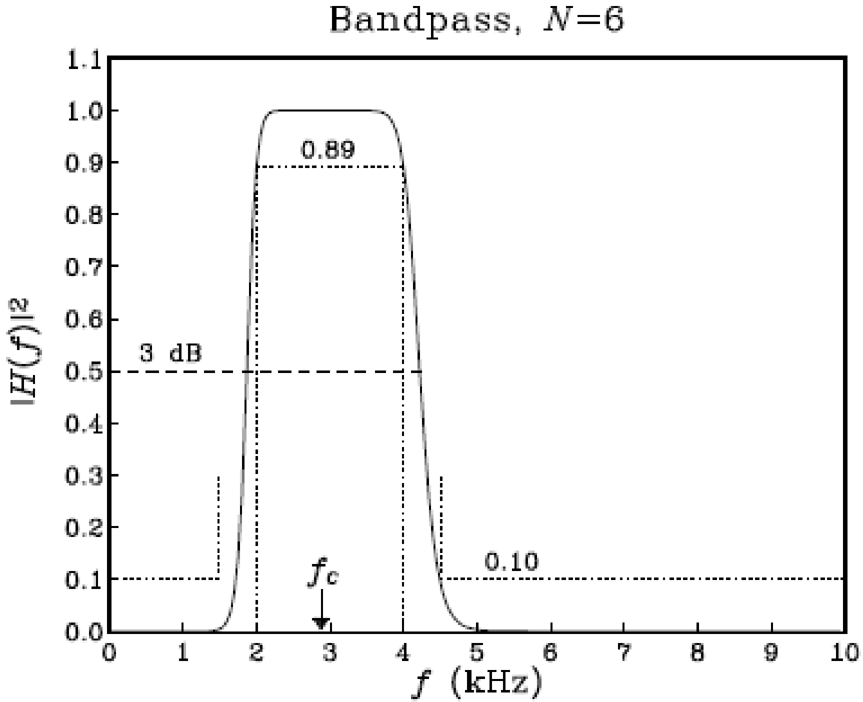
The left-right 3-dB frequencies are calculated to be: $f_{0a} = 1.8689$ kHz, $f_{0b} = 4.2206$ kHz. The coefficients of the three fourth-order sections of the digital filter are as followed, and can be computed by the I2SP MATLAB function **bpsbutt.m**:

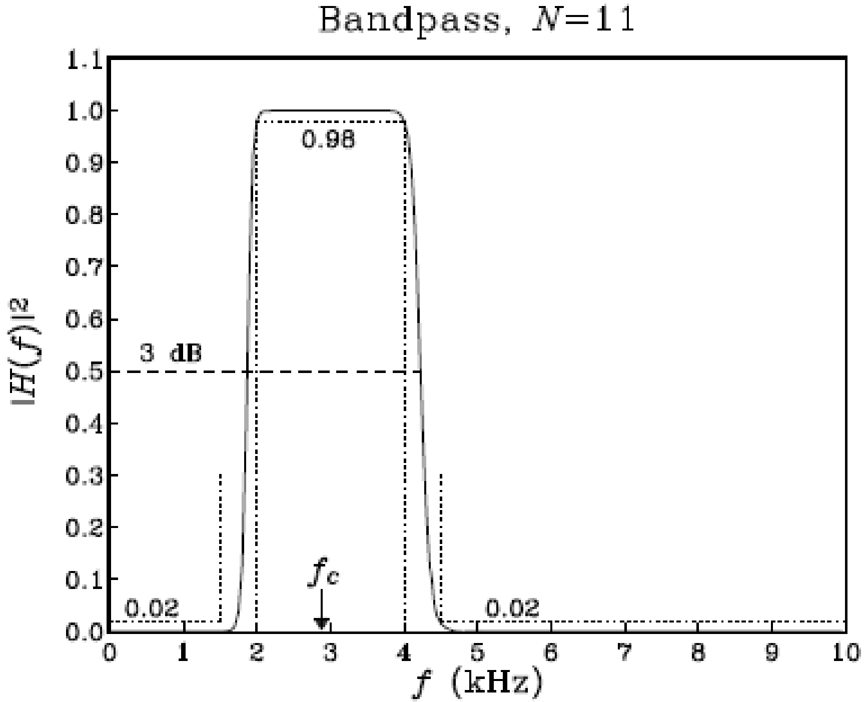
i	G_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}
1	0.1110	−2.0142	2.3906	−1.6473	0.7032
2	0.0883	−1.8551	1.9017	−1.0577	0.3549
3	0.0790	−1.7897	1.7009	−0.8154	0.2118

The magnitude response can be calculated from:

$$|H(\omega)|^2 = \frac{1}{1 + \left(\dfrac{\Omega}{\Omega_0}\right)^{2N}} = \frac{1}{1 + \left(\dfrac{c - \cos\omega}{\Omega_0 \sin\omega}\right)^{2N}}$$

The magnitude response is shown in the left graph of the figure below. The passband specifications are met exactly by design. Because the maximum stopband frequency was on the right, $\Omega_{\text{stop}} = |\Omega_{sb}|$, the right stopband specification is met stringently. The left stopband specification is more than required.





For the second set of specifications, we have $A_{\text{pass}} = -10\log_{10}(0.98) = 0.0877$ dB, and $A_{\text{stop}} = -10\log_{10}(0.02) = 16.9897$ dB and $\{\varepsilon_{\text{pass}}, \varepsilon_{\text{stop}}\} = \{0.1429, 7\}$. The design has the same c, Ω_{pass}, and Ω_{stop}, which lead to the Butterworth parameters:

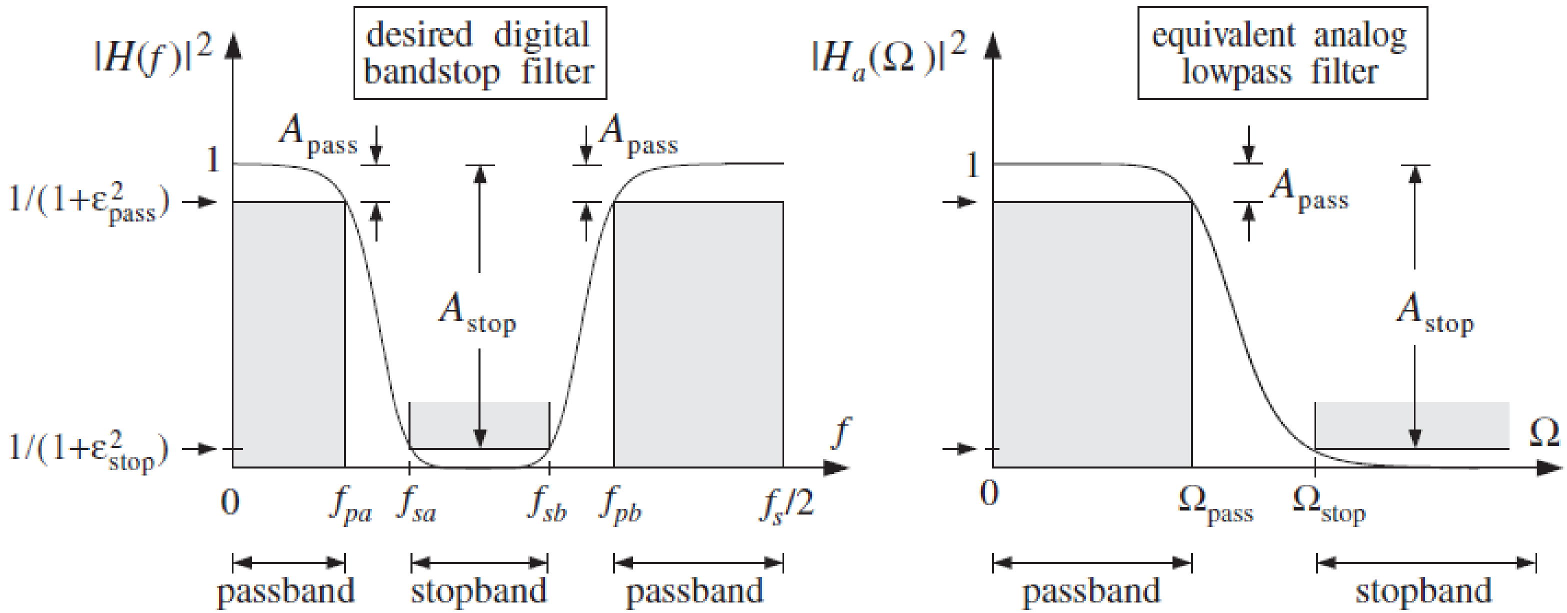
$$N_{\text{exact}} = 10.71, \quad N = 11, \quad \Omega_0 = 0.3878$$

The left and right 3-dB frequencies are now $f_{0a} = 1.8677$ kHz, $f_{0b} = 4.2228$ kHz. The digital filter coefficients of the second- and fourth-order sections are:

i	G_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}
0	0.2794	−0.8907	0.4411		
1	0.1193	−2.0690	2.5596	−1.8526	0.8249
2	0.1021	−1.9492	2.1915	−1.4083	0.5624
3	0.0907	−1.8694	1.9460	−1.1122	0.3874
4	0.0834	−1.8186	1.7900	−0.9239	0.2762
5	0.0794	−1.7904	1.7033	−0.8193	0.2144

Again, the right stopband specification is more stringently met than the left one. The "center" frequency of the passband is the same for both filters and can be obtained by inverting $\cos\omega_c = c$. In Hz, we have $f_c = f_s \arccos(c)/(2\pi) = 2.8793$ kHz. The magnitude response is normalized to unity at f_c.

Digital Bandstop Filters



The specifications are the quantities $\{f_{pa}, f_{pb}, f_{sa}, f_{sb}, A_{\text{pass}}, A_{\text{stop}}\}$, defining the frequency bands:

– the stop range $f_{sa} \leq f \leq f_{sb}$,

– the left passband $0 \leq f \leq f_{pa}$, and

– the right passband $f_{pb} \leq f \leq f_s/2$.

where we assumed that the passaband attenuations are the same in the two passbands, if not, we can work with the smallest of the two.

The bandstop version of the bilinear transformation and the corresponding frequency mapping are,

$$s = \frac{1 - z^{-2}}{1 - 2cz^{-1} + z^{-2}}, \qquad \Omega = \frac{\sin\omega}{\cos\omega - c}, \qquad \omega = \frac{2\pi f}{f_s}$$

The design steps are summarized as follows. First, we compute the digital frequencies in radians per sample:

$$\omega_{pa} = \frac{2\pi f_{pa}}{f_s}, \quad \omega_{pb} = \frac{2\pi f_{pb}}{f_s}, \quad \omega_{sa} = \frac{2\pi f_{sa}}{f_s}, \quad \omega_{sb} = \frac{2\pi f_{sb}}{f_s}$$

Then, we calculate c and Ω_{pass} by requiring:

$$-\Omega_{\text{pass}} = \frac{\sin\omega_{pa}}{\cos\omega_{pa} - c}, \quad \Omega_{\text{pass}} = \frac{\sin\omega_{pb}}{\cos\omega_{pb} - c}$$

which may be solved as follows:

$$c = \frac{\sin(\omega_{pa} + \omega_{pb})}{\sin\omega_{pa} + \sin\omega_{pb}}, \qquad \Omega_{\text{pass}} = \left|\frac{\sin\omega_{pb}}{\cos\omega_{pb} - c}\right|$$

Next, we compute the two possible stopbands:

$$\Omega_{sa} = \frac{\sin\omega_{sa}}{\cos\omega_{sa} - c}, \qquad \Omega_{sb} = \frac{\sin\omega_{sb}}{\cos\omega_{sb} - c}$$

and define:

$$\Omega_{\text{stop}} = \min\big(|\Omega_{sa}|, |\Omega_{sb}|\big)$$

Then, use the analog specifications $\{\Omega_{\text{pass}}, \Omega_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$ to compute the Butterworth parameters $\{N, \Omega_0\}$.

And finally, transform the analog filter sections into fourth-order sections,

$$H_i(z) = \left.\frac{1}{1 - 2\dfrac{s}{\Omega_0}\cos\theta_i + \dfrac{s^2}{\Omega_0^2}}\right|_{s=\frac{1-z^{-2}}{1-2cz^{-1}+z^{-2}}}$$

$$= \frac{G_i(1 - 2cz^{-1} + z^{-2})^2}{1 + a_{i1}z^{-1} + a_{i2}z^{-2} + a_{i3}z^{-3} + a_{i4}z^{-4}}$$

where the coefficients are given for $i = 1, 2, \ldots, K$:

$$G_i = \frac{\Omega_0^2}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}$$

$$a_{i1} = \frac{4c\Omega_0(\cos\theta_i - \Omega_0)}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2} \qquad a_{i2} = \frac{2(2c^2\Omega_0^2 + \Omega_0^2 - 1)}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}$$

$$a_{i3} = -\frac{4c\Omega_0(\cos\theta_i + \Omega_0)}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2} \qquad a_{i4} = \frac{1 + 2\Omega_0\cos\theta_i + \Omega_0^2}{1 - 2\Omega_0\cos\theta_i + {\Omega_0}^2}$$

If N is odd, we also have a second-order section in z:

$$H_0(z) = \frac{1}{1 + \dfrac{s}{\Omega_0}}\Bigg|_{s=\frac{1-z^{-2}}{1-2cz^{-1}+z^{-2}}} = \frac{G_0(1 - 2cz^{-1} + z^{-2})}{1 + a_{01}z^{-1} + a_{02}z^{-2}}$$

where

$$G_0 = \frac{\Omega_0}{1+\Omega_0}, \qquad a_{01} = -\frac{2c\Omega_0}{1+\Omega_0}, \qquad a_{02} = -\frac{1-\Omega_0}{1+\Omega_0}$$

Note that each section has zeros at $1 - 2cz^{-1} + z^{-2} = 0$, which correspond to the angles $\omega = \pm\omega_c$, where $\cos\omega_c = c$. The 3-dB frequencies at the edges of the passbands can be determined by solving for the positive solutions of the equations:

$$\frac{\sin\omega_0}{\cos\omega_0 - c} = \pm\Omega_0$$

which give:

$$\tan\left(\frac{\omega_{0a}}{2}\right) = \tan\left(\frac{\pi f_{0a}}{f_s}\right) = \frac{\sqrt{1 + \Omega_0^2(1 - c^2)} - 1}{\Omega_0(1 + c)}$$

$$\tan\left(\frac{\omega_{0b}}{2}\right) = \tan\left(\frac{\pi f_{0b}}{f_s}\right) = \frac{\sqrt{1 + \Omega_0^2(1 - c^2)} + 1}{\Omega_0(1 + c)}$$

Example:
Using the bilinear transformation and a lowpass analog Butterworth prototype, design a bandstop digital filter operating at a rate of 20 kHz and having left and right passband frequencies of 1.5 and 4.5 kHz, and left and right stopband frequencies of 2 and 4 kHz. The maximum passband attenuation is required to be 0.5 dB, and the minimum stopband attenuation 10 dB. Then, redesign it such that its magnitude response satisfies

$$1 \geq |H(f)|^2 \geq 0.98$$

in the passbands, and

$$|H(f)|^2 \leq 0.02$$

in the stopband.

Solution: The digital passband and stopband frequencies are:

$$\omega_{pa} = \frac{2\pi f_{pa}}{f_s} = \frac{2\pi \cdot 1.5}{20} = 0.15\pi, \quad \omega_{pb} = \frac{2\pi f_{pb}}{f_s} = \frac{2\pi \cdot 4.5}{20} = 0.45\pi$$

$$\omega_{sa} = \frac{2\pi f_{sa}}{f_s} = \frac{2\pi \cdot 2}{20} = 0.2\pi, \qquad \omega_{sb} = \frac{2\pi f_{sb}}{f_s} = \frac{2\pi \cdot 4}{20} = 0.4\pi$$

Then, we calculate c and Ω_{pass}:

$$c = \frac{\sin(\omega_{pa} + \omega_{pb})}{\sin\omega_{pa} + \sin\omega_{pb}} = 0.6597, \qquad \Omega_{\text{pass}} = \left|\frac{\sin\omega_{pb}}{\cos\omega_{pb} - c}\right| = 1.9626$$

Then, we calculate the stopband frequencies:

$$\Omega_{sa} = \frac{\sin\omega_{sa}}{\cos\omega_{sa} - c} = 3.9361, \qquad \Omega_{sb} = \frac{\sin\omega_{sb}}{\cos\omega_{sb} - c} = -2.7121$$

and define $\Omega_{\text{stop}} = \min\bigl(|\Omega_{sa}|, |\Omega_{sb}|\bigr) = 2.7121$. The analog filter parameters are:

$$N_{\text{exact}} = 6.65, \quad N = 7, \quad \Omega_0 = 2.2808$$

The left-right 3-dB frequencies are calculated to be $f_{0a} = 1.6198$ kHz, $f_{0b} = 4.2503$ kHz. The coefficients of the SOS and the three fourth-order sections of the digital filter are:

i	G_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}
0	0.6952	−0.9172	0.3904		
1	0.7208	−2.0876	2.4192	−1.7164	0.7187
2	0.5751	−1.9322	1.9301	−1.1026	0.3712
3	0.5045	−1.8570	1.6932	−0.8053	0.2029

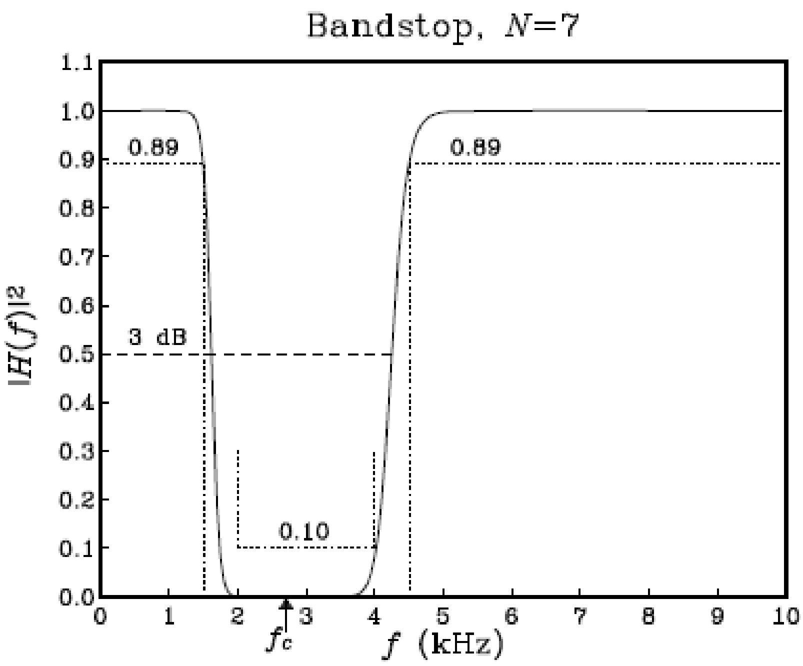
For the second set of specifications, we have $A_{\text{pass}} = -10\log_{10}(0.98) = 0.0877$ dB, and $A_{\text{stop}} = -10\log_{10}(0.02) = 16.9897$ dB. The design has the same c, Ω_{pass}, and Ω_{stop}, which lead to the Butterworth parameters:

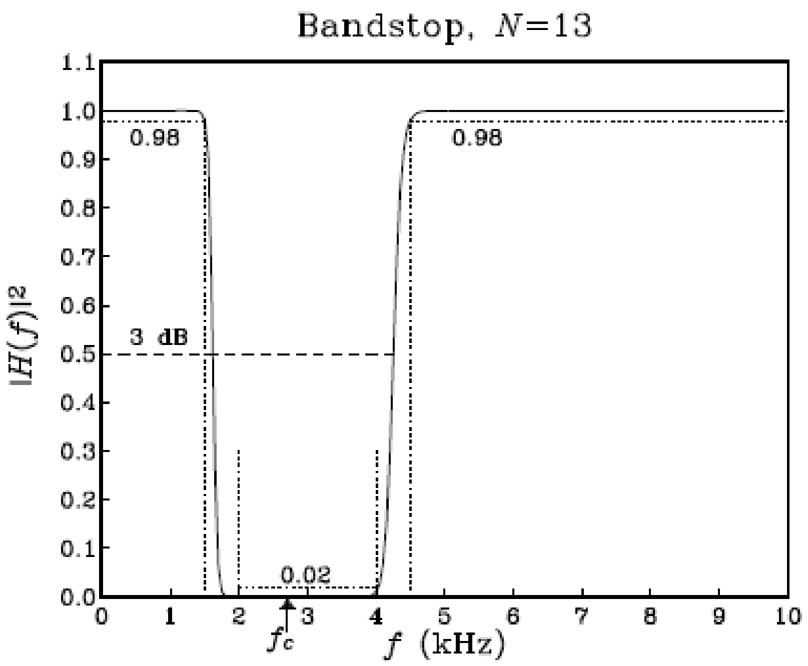
$$N_{\text{exact}} = 12.03, \quad N = 13, \quad \Omega_0 = 2.2795$$

The left-right 3-dB frequencies are now $f_{0a} = 1.6194$ kHz, $f_{0b} = 4.2512$ kHz. The filter coefficients of the second- and fourth-order sections are:

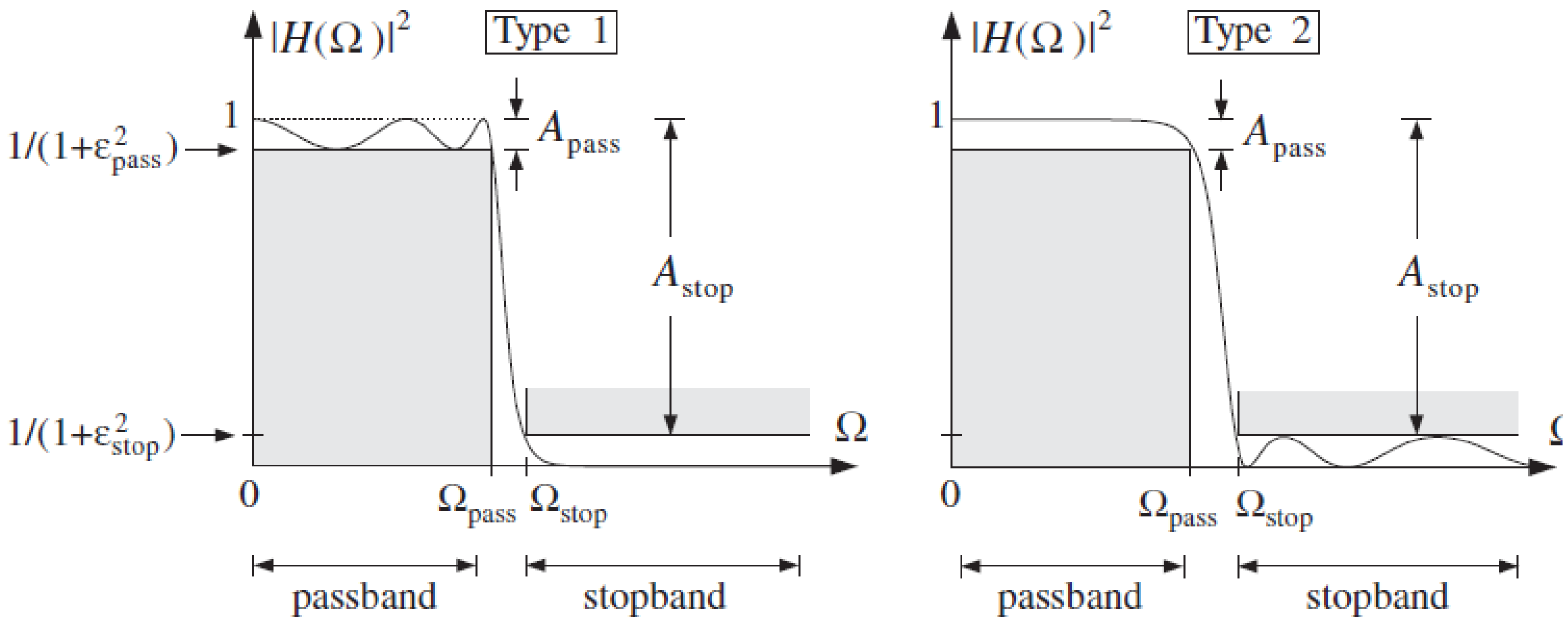
i	G_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}
0	0.6951	−0.9171	0.3902		
1	0.7703	−2.1401	2.5850	−1.9251	0.8371
2	0.6651	−2.0280	2.2319	−1.4820	0.5862
3	0.5914	−1.9495	1.9847	−1.1717	0.4105
4	0.5408	−1.8956	1.8148	−0.9584	0.2897
5	0.5078	−1.8604	1.7041	−0.8194	0.2110
6	0.4892	−1.8406	1.6415	−0.7410	0.1666

For both designs, the "center" notch frequency of the stopband can be obtained by inverting $\cos\omega_c = c$. In Hz, we have $f_c = f_s \arccos(c)/(2\pi) = 2.7069$ kHz. The magnitude responses of the designed filters are shown below.





Chebyshev Filter Design



Chebyshev filters come in two varieties. Type 1 has equiripple passband and monotonic stopband, and type 2, also known as inverse Chebyshev, has equiripple stopband and monotonic passband.

It is the equiripple property that is responsible for the narrower transition widths of these filters. For example, for the type 1 case, because the passband response is allowed to go slightly up near the edge of the passband, it can fall off more steeply.

The specifications of the filter are $\{\Omega_{\text{pass}}, \Omega_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$ and are obtained by prewarping the desired digital filter specifications using the appropriate bilinear transformation (lowpass, highpass, bandpass, or bandstop). Two important design parameters are the quantities $\{\varepsilon_{\text{pass}}, \varepsilon_{\text{stop}}\}$. The magnitude response squared of an Nth order Chebyshev filter is expressible in terms of these parameters as follows. For the type 1 case:

$$\boxed{|H(\Omega)|^2 = \frac{1}{1 + \varepsilon_{\text{pass}}^2 C_N^2\left(\dfrac{\Omega}{\Omega_{\text{pass}}}\right)}}$$

and, for the type 2 case:

$$\boxed{|H(\Omega)|^2 = \frac{C_N^2\left(\dfrac{\Omega_{\text{stop}}}{\Omega}\right)}{C_N^2\left(\dfrac{\Omega_{\text{stop}}}{\Omega}\right) + \varepsilon_{\text{stop}}^2}}$$

where $C_N(x)$ is the Chebyshev polynomial of degree N, defined by,

$$C_N(x) = \begin{cases} \cos\big(N\cos^{-1}(x)\big), & \text{if } |x| \le 1 \\ \cosh\big(N\cosh^{-1}(x)\big), & \text{if } |x| > 1 \end{cases}$$

Chebyshev polynomials can be understood by defining the angle $\theta = \cos^{-1} x$, so that $x = \cos\theta$ and $C_N(x) = \cos(N\theta)$.

When $|x| > 1$, the equation $x = \cos\theta$ requires θ to be imaginary, say $\theta = j\beta$, so that $x = \cos(j\beta) = \cosh(\beta)$ and

$$C_N(x) = \cos(N\theta) = \cos(Nj\beta) = \cosh(N\beta) = \cosh(N\cosh^{-1} x)$$

Using trigonometric identities, it can be shown that $\cos(N\theta)$ is expressible as an Nth order polynomial in $\cos\theta$, that is,

$$\cos(N\theta) = \sum_{i=0}^{N} c_i (\cos\theta)^i$$

The c_i are the coefficients of the Chebyshev polynomials:

$$C_N(x) = \sum_{i=0}^{N} c_i x^i$$

For example, we have $C_1(x) = \cos\theta = x$, and

$$\begin{aligned} \cos(2\theta) &= 2\cos^2\theta - 1 & & C_2(x) = 2x^2 - 1 \\ \cos(3\theta) &= 4\cos^3\theta - 3\cos\theta & \Rightarrow \quad & C_3(x) = 4x^3 - 3x \\ \cos(4\theta) &= 8\cos^4\theta - 8\cos^2\theta + 1 & & C_4(x) = 8x^4 - 8x^2 + 1 \end{aligned}$$

Next, we consider the details of the type 1 case. The argument of $C_N(x)$ is $x = \Omega/\Omega_{\text{pass}}$. Therefore, within the passband range $0 \le \Omega \le \Omega_{\text{pass}}$ we have $0 \le x \le 1$, which makes $C_N(x)$ oscillatory and results in the passband ripples.

Within the passband, the magnitude response remains bounded between the values 1 and $1/(1+\varepsilon_{\text{pass}}^2)$. At the edge of the passband, corresponding to $x = \Omega/\Omega_{\text{pass}} = 1$, we have $C_N(x) = 1$, giving the value $|H(\Omega_{\text{pass}})|^2 = 1/(1+\varepsilon_{\text{pass}}^2)$. The value at $\Omega = 0$ depends on N. Because $C_N(0)$ equals zero for odd N and unity for even N, we have:

$$|H(0)|^2 = 1 \quad (\text{odd } N), \qquad |H(0)|^2 = \frac{1}{1+\varepsilon_{\text{pass}}^2} \quad (\text{even } N)$$

The order N can be determined by imposing the stopband specification, that is, $|H(\Omega)|^2 \leq 1/(1+\varepsilon_{\text{stop}}^2)$ for $\Omega \geq \Omega_{\text{stop}}$. Because of the monotonicity of the stopband, this condition is equivalent to the stopband edge condition:

$$|H(\Omega_{\text{stop}})|^2 = \frac{1}{1+\varepsilon_{\text{stop}}^2}$$

and

$$\frac{1}{1+\varepsilon_{\text{pass}}^2 \cosh^2\big(N\cosh^{-1}(\Omega_{\text{stop}}/\Omega_{\text{pass}})\big)} = \frac{1}{1+\varepsilon_{\text{stop}}^2}$$

which gives:

$$\cosh\big(N\cosh^{-1}(\Omega_{\text{stop}}/\Omega_{\text{pass}})\big) = \varepsilon_{\text{stop}}/\varepsilon_{\text{pass}} \quad \Rightarrow \quad \cosh(N\cosh^{-1} w) = e$$

where we used the stopband to passband ratios:

$$e = \frac{\varepsilon_{\text{stop}}}{\varepsilon_{\text{pass}}} = \sqrt{\frac{10^{A_{\text{stop}}/10}-1}{10^{A_{\text{pass}}/10}-1}}\,, \quad w = \frac{\Omega_{\text{stop}}}{\Omega_{\text{pass}}}$$

Thus, solving for N, we find:

$$\boxed{N_{\text{exact}} = \frac{\cosh^{-1} e}{\cosh^{-1} w} = \frac{\ln\big(e+\sqrt{e^2-1}\big)}{\ln\big(w+\sqrt{w^2-1}\big)}}$$

The final value of N is obtained by rounding N_{exact} up to the next integer, that is, $N = \lceil N_{\text{exact}} \rceil$. As in the Butterworth case, increasing N slightly from its exact value results in a slightly better stopband than required, that is, $|H(\Omega_{\text{stop}})|^2 < 1/(1+\varepsilon_{\text{stop}}^2)$.

The 3-dB frequency can be calculated by requiring $|H(\Omega)|^2 = 1/2$, which can be solved to give:

$$\frac{1}{1+\varepsilon_{\text{pass}}^2 C_N^2(\Omega_{\text{3dB}}/\Omega_{\text{pass}})} = \frac{1}{2} \quad \Rightarrow \quad \cosh\bigl(N\cosh^{-1}(\Omega_{\text{3dB}}/\Omega_{\text{pass}})\bigr) = \frac{1}{\varepsilon_{\text{pass}}}$$

or,

$$\tan\left(\frac{\pi f_{\text{3dB}}}{f_s}\right) = \Omega_{\text{3dB}} = \Omega_{\text{pass}}\cosh\left(\frac{1}{N}\cosh^{-1}(\frac{1}{\varepsilon_{\text{pass}}})\right)$$

The transfer function $H(s)$ of the Chebyshev filter can be constructed by determining the left-hand-plane poles and pairing them in conjugate pairs to form second-order sections. These conjugate pairs are $\{s_i, s_i^*\}$, where,

$$s_i = \Omega_{\text{pass}} \sinh a \, \cos\theta_i + j\Omega_{\text{pass}} \cosh a \, \sin\theta_i, \quad i = 1, 2, \ldots, K$$

where $N = 2K$ or $N = 2K + 1$. In the odd case, there is also a real pole at

$$s_0 = -\Omega_{\text{pass}} \sinh a$$

where the parameter a is the solution of

$$\sinh(Na) = \frac{1}{\varepsilon_{\text{pass}}}$$

that is,

$$a = \frac{1}{N} \sinh^{-1}\left(\frac{1}{\varepsilon_{\text{pass}}}\right) = \frac{1}{N} \ln\left(\frac{1}{\varepsilon_{\text{pass}}} + \sqrt{\frac{1}{\varepsilon_{\text{pass}}^2} + 1}\right)$$

The angles θ_i are the *same* as the Butterworth angles,

$$\theta_i = \frac{\pi}{2N}(N - 1 + 2i), \quad i = 1, 2, \ldots, K$$

The 2nd quadrant values of these angles place the s_i in the left-hand s-plane.

The second-order sections are then:

$$H_i(s) = \frac{1}{\left(1 - \dfrac{s}{s_i}\right)\left(1 - \dfrac{s}{s_i^*}\right)} = \frac{|s_i|^2}{s^2 - (2\text{Re}s_i)s + |s_i|^2}$$

For convenience, we define the parameters:

$$\Omega_0 = \Omega_{\text{pass}} \sinh a, \qquad \Omega_i = \Omega_{\text{pass}} \sin\theta_i\,, \quad i = 1, 2, \dots, K$$

Then, we may express the second-order sections in the form:

$$H_i(s) = \frac{\Omega_0^2 + {\Omega_i}^2}{s^2 - 2\Omega_0 \cos\theta_i\, s + \Omega_0^2 + {\Omega_i}^2}\,, \quad i = 1, 2, \dots, K$$

The first-order factor $H_0(s)$ is defined by

$$H_0(s) = \begin{cases} \sqrt{\dfrac{1}{1 + \varepsilon_{\text{pass}}^2}} & \text{if } N \text{ is even, } N = 2K \\ \dfrac{\Omega_0}{s + \Omega_0} & \text{if } N \text{ is odd, } N = 2K + 1 \end{cases}$$

If N is odd, all filter sections are normalized to unity gain at DC. If N is even, the overall gain is $1/(1+\varepsilon_{\text{pass}}^2)^{1/2}$. It follows that the overall transfer function will be the cascade:

$$H(s) = H_0(s)H_1(s)H_2(s)\cdots H_K(s)$$

Once the analog transfer function is constructed, each second-order section may be transformed into a digital second-order section by the appropriate bilinear transformation. For example, applying the lowpass version of the bilinear transformation $s = (1-z^{-1})/(1+z^{-1})$, we find the digital transfer function:

$$H(z) = H_0(z)H_1(z)H_2(z)\cdots H_K(z)$$

where $H_i(z)$ will have the form:

$$H_i(z) = \frac{G_i(1+z^{-1})^2}{1+a_{i1}z^{-1}+a_{i2}z^{-2}}\,, \quad i = 1, 2, \ldots, K$$

and the coefficients are computed by

$$G_i = \frac{\Omega_0^2 + {\Omega_i}^2}{1 - 2\Omega_0 \cos\theta_i + \Omega_0^2 + {\Omega_i}^2}$$

$$a_{i1} = \frac{2(\Omega_0^2 + {\Omega_i}^2 - 1)}{1 - 2\Omega_0 \cos\theta_i + \Omega_0^2 + {\Omega_i}^2}$$

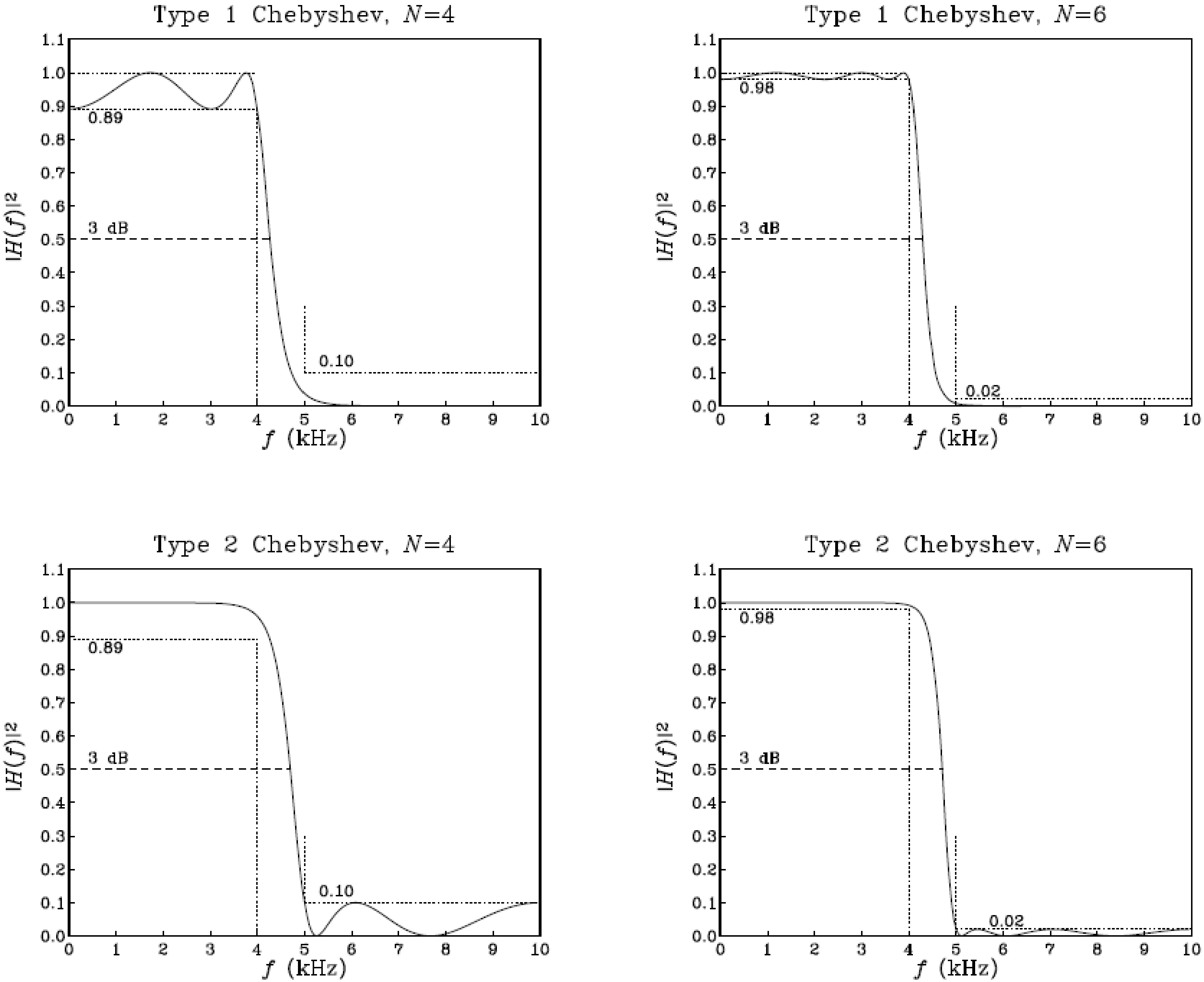
$$a_{i2} = \frac{1 + 2\Omega_0 \cos\theta_i + \Omega_0^2 + {\Omega_i}^2}{1 - 2\Omega_0 \cos\theta_i + \Omega_0^2 + {\Omega_i}^2}$$

The first-order factor is given by

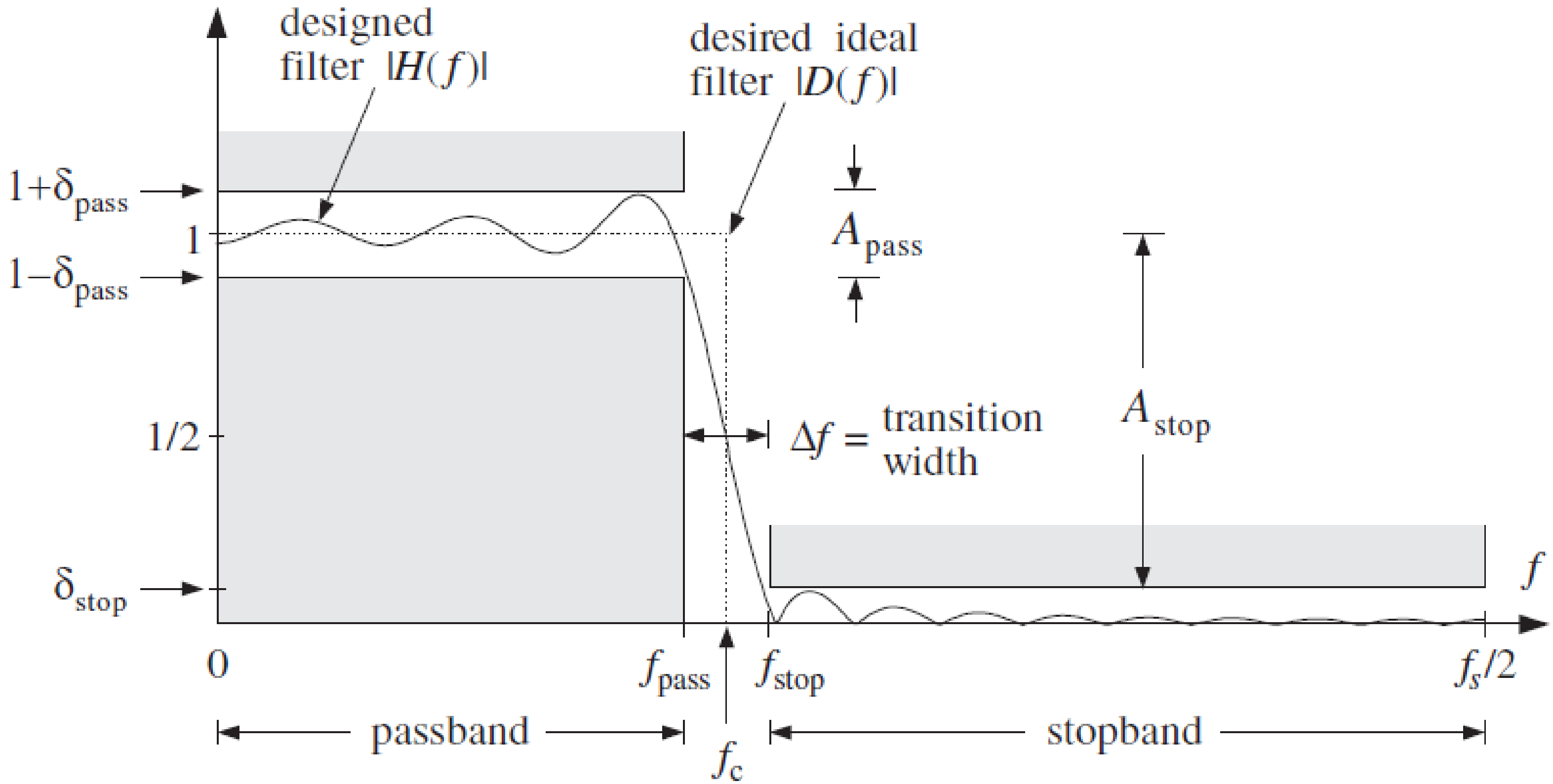
$$H_0(z) = \begin{cases} \sqrt{\dfrac{1}{1+\varepsilon_{\text{pass}}^2}} & \text{if } N \text{ is even} \\[2ex] \dfrac{G_0(1+z^{-1})}{1+a_{01}z^{-1}} & \text{if } N \text{ is odd} \end{cases}$$

where

$$G_0 = \frac{\Omega_0}{\Omega_0 + 1}, \quad a_{01} = \frac{\Omega_0 - 1}{\Omega_0 + 1}$$


Type 1 Chebyshev, N=4
Type 1 Chebyshev, N=6
Type 2 Chebyshev, N=4
Type 2 Chebyshev, N=6
|H(f)|²
f (kHz)
3 dB
0.89
0.10
0.98
0.02

FIR Digital Filter Design



The FIR filter design problem is the problem of determining the filter length N and the finite impulse response coefficient vector, $\mathbf{h} = [h_0, h_1, \ldots, h_{N-1}]$, of an FIR filter that meets *prescribed* frequency response specifications.

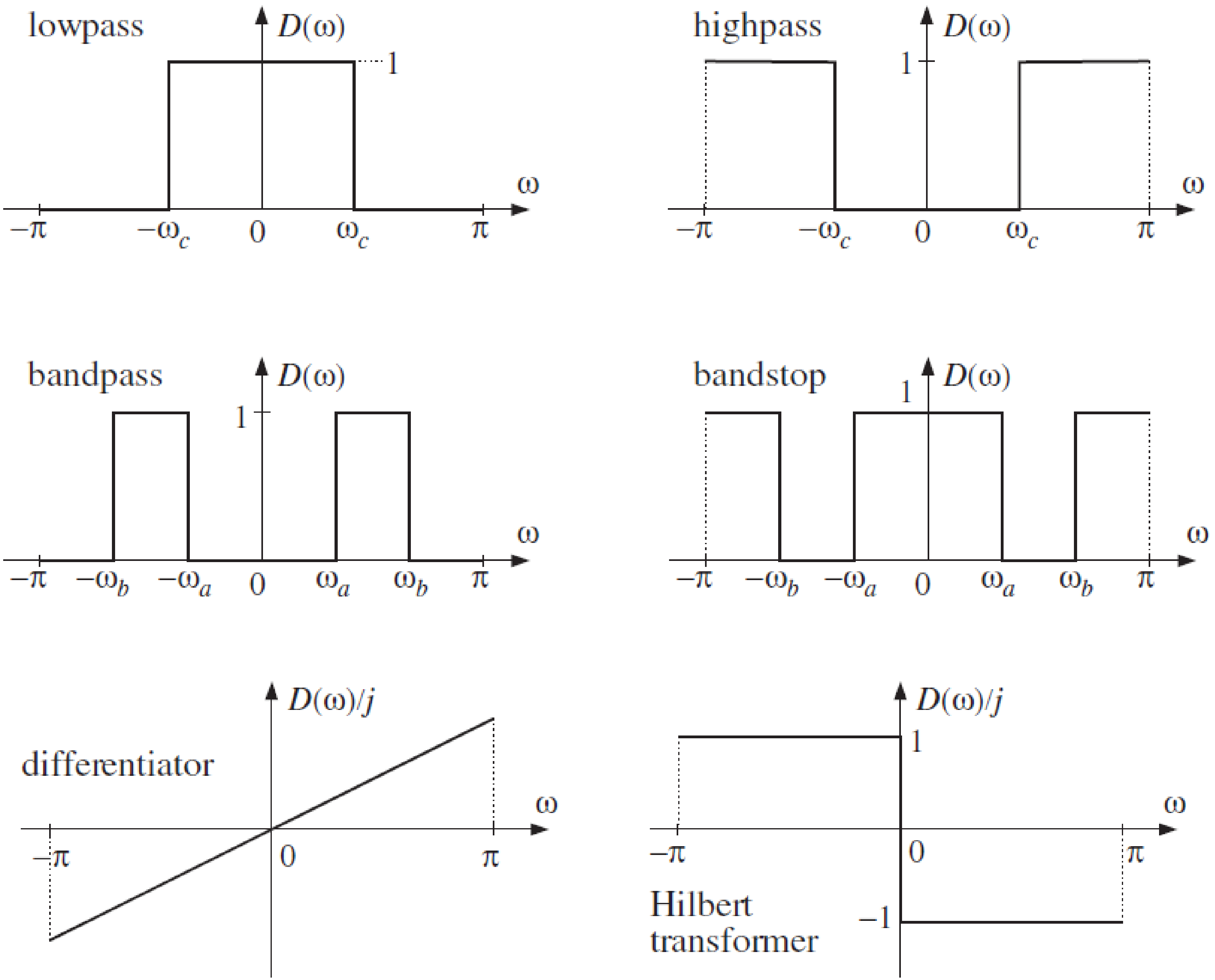
The two main advantages of FIR filters are their linear phase property and their guaranteed stability because of the absence of poles. Their potential disadvantage is that the requirement of sharp filter specifications can lead to long filter lengths N, consequently increasing their computational cost.

The main advantages of IIR filters are their low computational cost and their *efficient implementation* in cascade of second-order sections. Their main disadvantage is the potential for instabilities introduced when the quantization of the coefficients pushes the poles outside the unit circle. For IIR filters, linear phase cannot be achieved exactly over the entire Nyquist interval, but it can be achieved *approximately* over the relevant passband of the filter, for example, using Bessel filter designs.

Window Method

The window method is one of the simplest methods of designing FIR digital filters. It is well suited for designing filters with *simple* frequency response shapes, such as ideal lowpass filters. Some typical filter shapes that can be designed are shown below. For arbitrary shapes, a variant of the method, known as frequency sampling method, may be used.

Window Method



A given desired ideal frequency response, say $D(\omega)$, being periodic in ω with period 2π, need only be specified over one complete Nyquist interval $-\pi \leq \omega \leq \pi$. The corresponding impulse response, say $d(k)$, is related to $D(\omega)$ by the DTFT and inverse DTFT relationships:

$$D(\omega) = \sum_{k=-\infty}^{\infty} d(k)e^{-j\omega k} \quad \Leftrightarrow \quad d(k) = \int_{-\pi}^{\pi} D(\omega)e^{j\omega k}\,\frac{d\omega}{2\pi}$$

In general, the impulse response $d(k)$ will be *double-sided and infinite*. For many ideal filter shapes, the ω-integration can be done in closed form. For example, for the *lowpass filter* shown above, the quantity $D(\omega)$ is defined over the Nyquist interval by

$$D(\omega) = \begin{cases} 1, & \text{if} \quad |\omega| \leq \omega_c \\ 0, & \text{if} \quad \omega_c < |\omega| \leq \pi \end{cases}$$

Therefore, the inverse DTFT formula gives,

$$d(k) = \int_{-\pi}^{\pi} D(\omega)e^{j\omega k}\,\frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega k}\,\frac{d\omega}{2\pi}$$

$$= \left[\frac{e^{j\omega k}}{2\pi jk}\right]_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c k} - e^{-j\omega_c k}}{2\pi jk}$$

which simplifies into,

$$\text{(lowpass filter)} \qquad d(k) = \frac{\sin(\omega_c k)}{\pi k}\,, \qquad -\infty < k < \infty$$

For computational purposes, the case $k = 0$ may be handled separately. Taking the limit $k \to 0$, we have,

$$d(0) = \frac{\omega_c}{\pi}$$

Similarly, we find the impulse responses of the ideal highpass, bandpass, and bandstop filters, defined over $-\infty < k < \infty$,

$$\text{(highpass filter)} \qquad d(k) = \delta(k) - \frac{\sin(\omega_c k)}{\pi k}$$

$$\text{(bandpass filter)} \qquad d(k) = \frac{\sin(\omega_b k) - \sin(\omega_a k)}{\pi k}$$

$$\text{(bandstop filter)} \qquad d(k) = \delta(k) - \frac{\sin(\omega_b k) - \sin(\omega_a k)}{\pi k}$$

Note that for the same values of the cutoff frequencies ω_c, ω_a, ω_b, the lowpass/highpass and bandpass/bandstop filters are *complementary*, that is, their impulse responses add up to a unit impulse $\delta(k)$ and their frequency responses add up to unity (as can also be seen by inspecting the above figure,

$$d_{LP}(k) + d_{HP}(k) = \delta(k) \quad \Leftrightarrow \quad D_{LP}(\omega) + D_{HP}(\omega) = 1$$

$$d_{BP}(k) + d_{BS}(k) = \delta(k) \quad \Leftrightarrow \quad D_{BP}(\omega) + D_{BS}(\omega) = 1$$

See I2SP Ch.10 for some audio graphic equalizer designs that exploit this property.

The ideal differentiator filter has frequency response, $D(\omega) = j\omega$, defined over the Nyquist interval. The ideal Hilbert transformer response can be expressed compactly as, $D(\omega) = -j\,\mathrm{sign}(\omega)$, where $\mathrm{sign}(\omega)$ is the signum function which is equal to ± 1 depending on the algebraic sign of its argument. The ω-integrations result in the ideal impulse responses:

$$\text{(differentiator)} \qquad d(k) = \frac{\cos(\pi k)}{k} - \frac{\sin(\pi k)}{\pi k^2}$$

$$\text{(Hilbert transformer)} \qquad d(k) = \frac{1 - \cos(\pi k)}{\pi k}$$

Both filters have $d(0) = 0$, as can be verified by carefully taking the limit $k \to 0$.

Both impulse responses $d(k)$ are real-valued and odd (antisymmetric) functions of k. By contrast, the LP/HP/BP/BS ideal filters all have impulse responses that are real and even (symmetric) in k. We will refer to the two classes of filters of as the symmetric and antisymmetric classes.

In the frequency domain, the symmetric types are characterized by a frequency response $D(\omega)$ which is real and even in ω; the antisymmetric ones have $D(\omega)$ which is imaginary and odd in ω. One of the main consequences of these frequency properties is the linear phase property of the window designs.

To summarize, the designs fall into two classes, both of which result in linear phase filters.

(a) Symmetric

$$\boxed{d(k) \text{ is real \& even in } k} \quad \Leftrightarrow \quad \boxed{D(\omega) \text{ is real \& odd in } \omega}$$

(b) Antisymmetric

$$\boxed{d(k) \text{ is real \& odd in } k} \quad \Leftrightarrow \quad \boxed{D(\omega) \text{ is imaginary \& odd in } \omega}$$

Rectangular Window

The window method consists of truncating, or rectangularly windowing, the double-sided $d(k)$ to a finite length. For example, we may keep only the $(2M+1)$ coefficients:

$$\boxed{d(k) = \int_{-\pi}^{\pi} D(\omega)e^{j\omega k}\,\frac{d\omega}{2\pi}\,, \qquad -M \le k \le M}$$

Because the coefficients are taken symmetrically for positive and negative k's, the total number of coefficients will be *odd*, that is, $N = 2M+1$ (even values of N are also possible, but not discussed here).

The resulting N-dimensional coefficient vector is the FIR impulse response approximating the infinite ideal response:

$$\mathbf{d} = [d_{-M}, \ldots, d_{-1}, d_0, d_1, \ldots, d_M]$$

The time origin $k = 0$ is at the middle d_0 of this vector. To make the filter causal we may shift the time origin to the left of the vector and re-index the entries accordingly:

$$\mathbf{h} = \mathbf{d} = [h_0, \ldots, h_{M-1}, h_M, h_{M+1}, \ldots, h_{2M}]$$

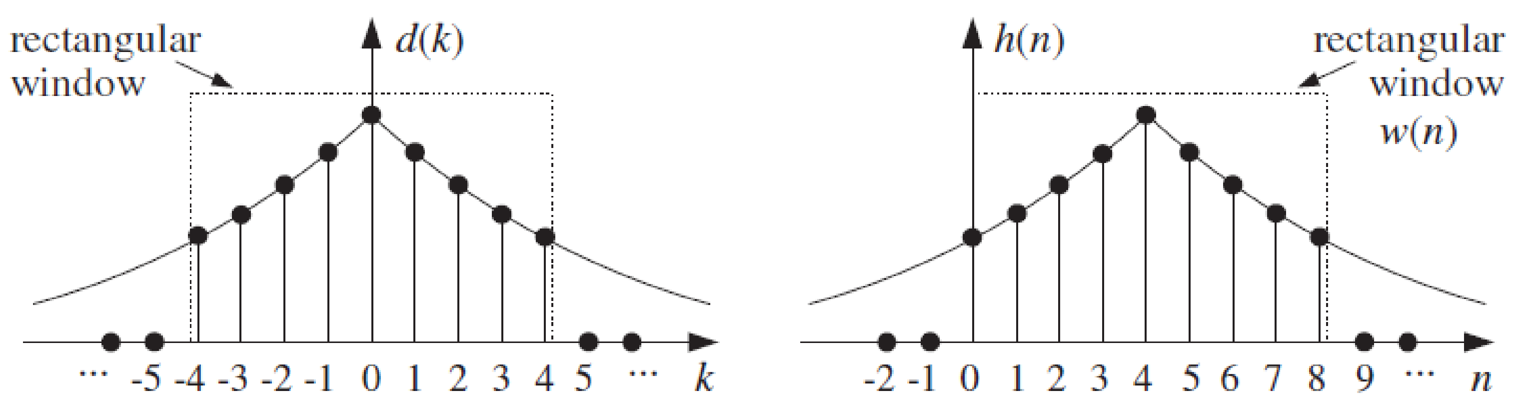
where we defined $h_0 = d_{-M}$, $h_1 = d_{-M+1}$, …, $h_M = d_0$, …, $h_{2M} = d_M$.

Thus, the vectors **d** and **h** are the same, with the understanding that **d**'s origin is in its middle and **h**'s at its left. The definition of **h** may be thought of as time-delaying the double-sided sequence $d(k)$, $-M \le k \le M$, by M time units to make it causal:

$$h(n) = d(n - M), \qquad n = 0, 1, \ldots, N - 1$$

The operations of windowing and delaying are shown below. To summarize, the steps of the *rectangular window method* are simply:

1. Pick an odd length $N = 2M + 1$, and let $M = (N - 1)/2$.
2. Calculate the N ideal coefficients $d(k)$, and
3. Make them causal by the delay operation.



For example, the length-N approximation to the ideal lowpass filter is,

$$h(n) = d(n-M) = \frac{\sin\big(\omega_c(n-M)\big)}{\pi(n-M)}\,, \quad n = 0, \ldots, M, \ldots, N-1$$

where we may calculate separately, $h(M) = d(0) = \omega_c/\pi$.

Example:
Determine the length-11, rectangularly windowed impulse response that approximates (a) an ideal lowpass filter of cutoff frequency $\omega_c = \pi/4$, (b) the ideal differentiator filter, and (c) the ideal Hilbert transformer filter.

Solution:
With $N = 11$, we have $M = (N-1)/2 = 5$. For the lowpass filter, we evaluate,

$$d(k) = \frac{\sin(\pi k/4)}{\pi k}\,, \quad \text{for } -5 \leq k \leq 5$$

and obtain the numerical values:

$$\mathbf{h} = \mathbf{d} = \left[-\frac{\sqrt{2}}{10\pi}, 0, \frac{\sqrt{2}}{6\pi}, \frac{1}{2\pi}, \frac{\sqrt{2}}{2\pi}, \frac{1}{4}, \frac{\sqrt{2}}{2\pi}, \frac{1}{2\pi}, \frac{\sqrt{2}}{6\pi}, 0, -\frac{\sqrt{2}}{10\pi}\right]$$

For the differentiator filter, the second term, $\sin(\pi k)/\pi k^2$, vanishes for all values $k \neq 0$. Therefore, we find:

$$\mathbf{h} = \mathbf{d} = \left[\frac{1}{5},\ -\frac{1}{4},\ \frac{1}{3},\ -\frac{1}{2},\ 1,\ 0,\ -1,\ \frac{1}{2},\ -\frac{1}{3},\ \frac{1}{4},\ -\frac{1}{5}\right]$$

And, for the Hilbert transformer:

$$\mathbf{h} = \mathbf{d} = \left[-\frac{2}{5\pi},\ 0,\ -\frac{2}{3\pi},\ 0,\ -\frac{2}{\pi},\ 0,\ \frac{2}{\pi},\ 0,\ \frac{2}{3\pi},\ 0,\ \frac{2}{5\pi}\right]$$

Note that the lowpass filter's impulse response is symmetric about its middle, whereas the differentiator's and Hilbert transformer's are antisymmetric. Note also that because of the presence of the factor, $1 - \cos(\pi k)$, every other entry of the Hilbert transformer vanishes.

In the frequency domain, the FIR approximation to $D(\omega)$ is equivalent to truncating the DTFT Fourier series expansion to the finite sum:

$$\boxed{\widehat{D}(\omega) = \sum_{k=-M}^{M} d(k)e^{-j\omega k}}$$

Replacing $z = e^{j\omega}$, we may also write it as the double-sided z-transform:

$$\widehat{D}(z) = \sum_{k=-M}^{M} d(k)z^{-k}$$

The final length-N causal filter will have transfer function:

$$\boxed{H(z) = z^{-M}\widehat{D}(z) = z^{-M}\sum_{k=-M}^{M} d(k)z^{-k}}$$

and frequency response:

$$\boxed{H(\omega) = e^{-j\omega M}\widehat{D}(\omega) = e^{-j\omega M}\sum_{k=-M}^{M} d(k)e^{-j\omega k}}$$

Example:

To illustrate the definition of $H(z)$, consider a case with $N = 7$ and $M = (N-1)/2 = 3$. Let the FIR filter weights be

$$\mathbf{d} = [d_{-3}, d_{-2}, d_{-1}, d_0, d_1, d_2, d_3]$$

with truncated z-transform:

$$\widehat{D}(z) = d_{-3}z^3 + d_{-2}z^2 + d_{-1}z + d_0 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}$$

Delaying it by $M = 3$, we get the causal transfer function:

$$\begin{aligned}
H(z) &= z^{-3}\widehat{D}(z) \\
&= z^{-3}\big(d_{-3}z^3 + d_{-2}z^2 + d_{-1}z + d_0 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}\big) \\
&= d_{-3} + d_{-2}z^{-1} + d_{-1}z^{-2} + d_0 z^{-3} + d_1 z^{-4} + d_2 z^{-5} + d_3 z^{-6} \\
&= h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + h_4 z^{-4} + h_5 z^{-5} + h_6 z^{-6}
\end{aligned}$$

where we defined $h(n) = d(n-3)$, $n = 0, 1, 2, 3, 4, 5, 6$.

The linear phase property of the window design is a direct consequence of the delaying operation and the fact that the truncated $\widehat{D}(\omega)$ has the same symmetry/antisymmetry properties as $D(\omega)$.

Thus, in the *symmetric* case, $\widehat{D}(\omega)$ will be real and even in ω. It follows that the designed FIR filter will have linear phase, arising essentially from the delay factor $e^{-j\omega M}$. More precisely, we may write the real factor $\widehat{D}(\omega)$ in terms of its positive magnitude and its sign:

$$\widehat{D}(\omega) = \mathrm{sign}\big(\widehat{D}(\omega)\big)\,|\widehat{D}(\omega)| = e^{j\pi\beta(\omega)}\,|\widehat{D}(\omega)|$$

where $\beta(\omega) = \big[1 - \mathrm{sign}\big(\widehat{D}(\omega)\big)\big]/2$, which is zero or one depending on the sign of $\widehat{D}(\omega)$. It follows that $H(\omega)$ will be:

$$H(\omega) = e^{-j\omega M}\,\widehat{D}(\omega) = e^{-j\omega M + j\pi\beta(\omega)}\,|\widehat{D}(\omega)|$$

Therefore, its magnitude and phase responses will be:

$$|H(\omega)| = |\widehat{D}(\omega)|\,, \quad \arg H(\omega) = -\omega M + \pi\,\beta(\omega)$$

making the phase response piece-wise linear in ω with 180^{o} jumps at those ω where $\widehat{D}(\omega)$ changes sign.

For the *antisymmetric* case, $\widehat{D}(\omega)$ will be pure imaginary, that is, of the form $\widehat{D}(\omega) = jA(\omega)$, with real-valued $A(\omega)$. The factor j may be made into a phase by writing it as $j = e^{j\pi/2}$. Thus, we have,

$$H(\omega) = e^{-j\omega M}\widehat{D}(\omega) = e^{-j\omega M}e^{j\pi/2}A(\omega) = e^{-j\omega M}e^{j\pi/2}e^{j\pi\alpha(\omega)}\,|A(\omega)|$$

where $\alpha(\omega) = \big[1 - \mathrm{sign}\big(A(\omega)\big)\big]/2$, which gives for the magnitude and phase responses:

$$|H(\omega)| = |A(\omega)|\,, \quad \arg H(\omega) = -\omega M + \frac{\pi}{2} + \pi\,\alpha(\omega)$$

How good is the rectangular window design? How well does the truncated $\widehat{D}(\omega)$ represent the desired response $D(\omega)$? In other words, how good is the approximation $\widehat{D}(\omega) \simeq D(\omega)$?

Intuitively one would expect that $\widehat{D}(\omega) \to D(\omega)$ as N increases. This is true for any ω which is a point of continuity of $D(\omega)$, but it fails at points of *discontinuity*, such as at the transition edges from passband to stopband. Around these edges one encounters the celebrated Gibbs phenomenon of Fourier series, which causes the approximation to be bad regardless of how large N is.

To illustrate the nature of the approximation $\widehat{D}(\omega) \simeq D(\omega)$, we consider the design of an ideal lowpass filter of cutoff frequency $\omega_c = 0.3\pi$, approximated by a rectangularly windowed response of length $N = 41$ and then by another one of length $N = 121$.

For the case $N = 41$, we have $M = (N-1)/2 = 20$. The designed impulse response is given by,

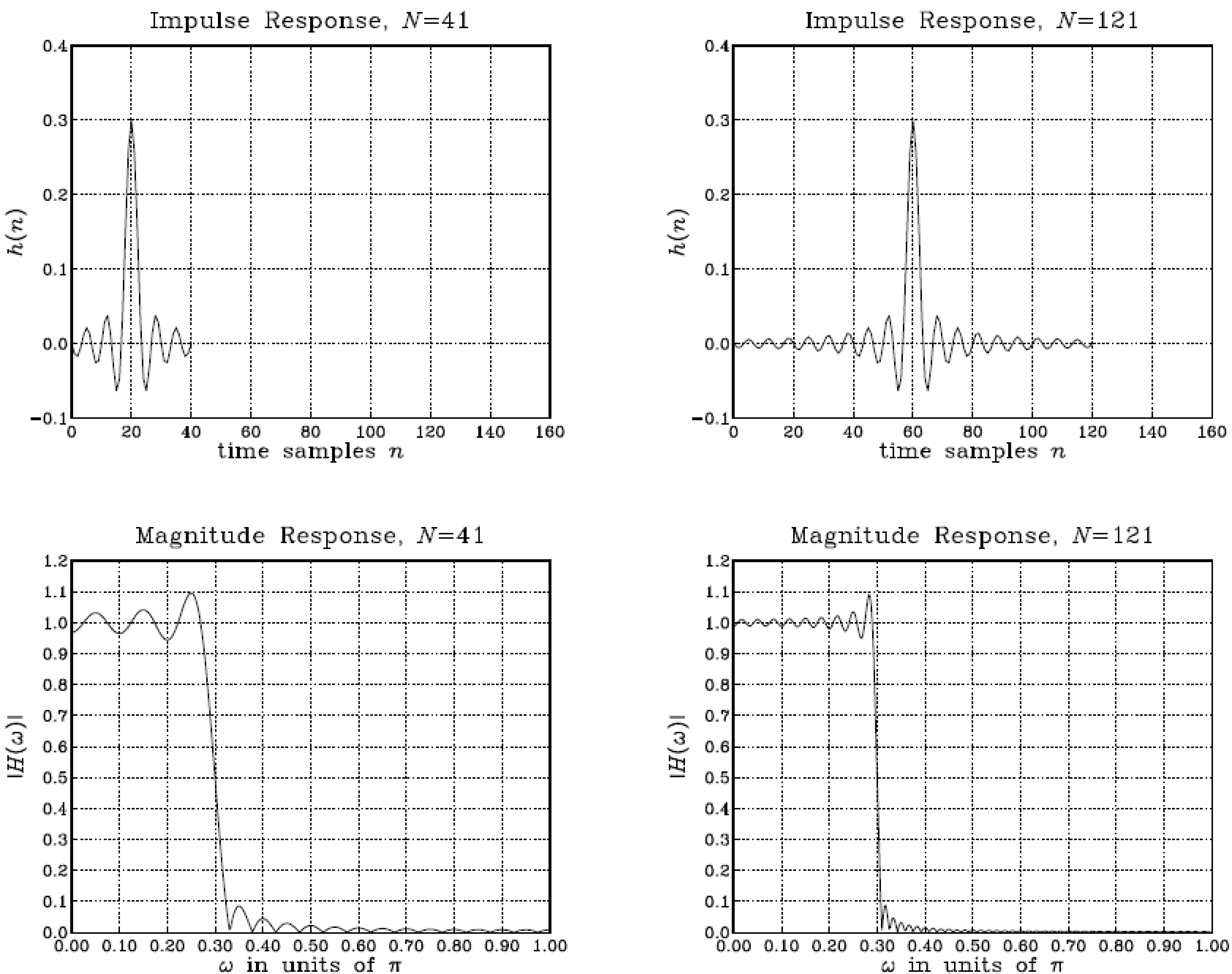
$$h(n) = d(n-20) = \frac{\sin\big(0.3\pi(n-20)\big)}{\pi(n-20)}\,, \qquad n = 0, 1, \ldots, 40$$

and in particular, $h(20) = d(0) = \omega_c/\pi = 0.3$. The second design has $N = 121$ and $M = 60$. Its impulse response is, with $h(60) = d(0) = 0.3$:

$$h(n) = d(n-60) = \frac{\sin\big(0.3\pi(n-60)\big)}{\pi(n-60)}\,, \qquad n = 0, 1, \ldots, 120$$

The two impulse responses are plotted below. Note that the portion of the second response extending ± 20 samples around the central peak at $n = 60$ coincides numerically with the first response. The corresponding magnitude responses are also shown below. They were computed by evaluating,

$$|H(\omega)| = \left|\sum_{n=0}^{N-1} h(n)e^{-j\omega n}\right|$$



Impulse Response, N=41
$h(n)$
time samples n
Impulse Response, N=121
$h(n)$
time samples n
Magnitude Response, N=41
$|H(\omega)|$
ω in units of π
Magnitude Response, N=121
$|H(\omega)|$
ω in units of π

The length-N impulse response $h(n)$ may be thought of formally as the rectangularly windowed double-sided sequence defined by

$$\boxed{h(n) = w(n)d(n-M)}\,, \qquad -\infty < n < \infty$$

where $w(n)$ is the length-N rectangular window. In the frequency domain, this translates to the convolution of the corresponding DTFTs,

$$H(\omega) = \int_{-\pi}^{\pi} W(\omega - \omega')e^{-j\omega' M} D(\omega')\,\frac{d\omega'}{2\pi}$$

where the $e^{-j\omega' M}$ arises from the delay in $d(n-M)$. The DTFT $W(\omega)$ of the (causal) rectangular window $w(n)$ is, with $N = 2M+1$,

$$W(\omega) = \sum_{n=0}^{2M} e^{-j\omega n} = \frac{1 - e^{-(2M+1)j\omega}}{1 - e^{-j\omega}} = e^{-jM\omega}\,\frac{\sin(N\omega/2)}{\sin(\omega/2)}$$

Thus, the designed filter $H(\omega)$ will be a smeared version of the desired shape $D(\omega)$. In particular, for the ideal lowpass case, because $D(\omega')$ is nonzero and unity only over the subinterval $-\omega_c \le \omega' \le \omega_c$, the frequency convolution integral becomes:

$$H(\omega) = \int_{-\omega_c}^{\omega_c} W(\omega - \omega')e^{-j\omega' M} \frac{d\omega'}{2\pi}$$

The ripples in the frequency response $H(\omega)$ arise from the (integrated) ripples of the rectangular window spectrum $W(\omega)$. As N increases, we observe three effects:

1. For ω's that lie well within the passband or stopband (i.e., points of continuity), the ripple size decreases as N increases, resulting in flatter passband and stopband. For such ω, we have

$$\widehat{D}(\omega) \to D(\omega) \quad \text{as } N \to \infty$$

2. The transition width decreases with increasing N. Note also that for any N, the windowed response $H(\omega)$ is always equal to 0.5 at the cutoff frequency $\omega = \omega_c$. (This is a standard property of Fourier series.)

3. The largest ripples tend to cluster near the passband-to-stopband discontinuity (from both sides) and do not get smaller with N. Instead, their size remains approximately *constant*, about 8.9 percent, independent of N. Eventually, as $N \to \infty$, these ripples get squeezed onto the discontinuity at $\omega = \omega_c$, occupying a set of measure zero. This behavior is the Gibbs phenomenon.

Explanation of the Gibbs Phenomenon

Let us start with symmetric rectangular window, w_k, $-M \le k \le M$ The windowed filter coefficients are,

$$\hat{d}_k = w_k d_k = \begin{cases} d_k\,, & -M \le k \le M \\ 0\,, & \text{otherwise} \end{cases}$$

with corresponding approximate and exact frequency responses,

$$\widehat{D}(\omega) = \sum_{k=-M}^{M} d_k e^{-j\omega k} = \sum_{k=-\infty}^{\infty} w_k d_k e^{-j\omega k}$$

$$D(\omega) = \sum_{k=-\infty}^{\infty} d_k e^{-j\omega k}$$

which are related by the convolution property,

$$\widehat{D}(\omega) = \int_{-\pi}^{\pi} W(\omega - \omega') D(\omega') \frac{d\omega'}{2\pi}$$

where the DTFT of the symmetric window is the undelayed version of the above causal one, where $N = 2M + 1$,

$$W(\omega) = \frac{\sin(N\omega/2)}{\sin(\omega/2)}$$

For even small values of N, such as $N = 7$, we have the approximation,

$$W(\omega) = \frac{\sin(N\omega/2)}{\sin(\omega/2)} \approx \frac{\sin(N\omega/2)}{\omega/2}$$

so that the $\widehat{D}(\omega)$ can be expressed as,

$$\widehat{D}(\omega) = \int_{-\pi}^{\pi} \frac{\sin(N(\omega - \omega')/2)}{\pi(\omega - \omega')}\, D(\omega')\, d\omega'$$

Next, consider an ideal lowpass filter with cutoff frequency ω_c,

$$D(\omega) = \begin{cases} 1\,, & |\omega| < \omega_c \\ 0.5\,, & \omega = \pm\omega_c \\ 0\,, & |\omega| > \omega_c \end{cases}$$

It can be expressed either in terms of the Heaviside unit-step function or in terms of the signum function, as follows,

$$D(\omega) = u(\omega + \omega_c) - u(\omega - \omega_c)$$

$$D(\omega) = \frac{1}{2}\,\text{sign}\,(\omega + \omega_c) - \frac{1}{2}\,\text{sign}\,(\omega - \omega_c)$$

which is a consequence of the relationship,

$$u(x) = \frac{1}{2} + \frac{1}{2}\,\text{sign}(x)$$

For such ideal lowpass filter, the approximation $\widehat{D}(\omega)$ becomes,

$$\widehat{D}(\omega) = \int_{-\omega_c}^{\omega_c} \frac{\sin(N(\omega - \omega')/2)}{\pi(\omega - \omega')}\, d\omega'$$

Introducing the sine-integral, $\text{Si}(x)$, function,

$$\text{Si}(x) = \int_0^x \frac{\sin v}{v}\, dv$$

we may express $\widehat{D}(\omega)$ as the difference,

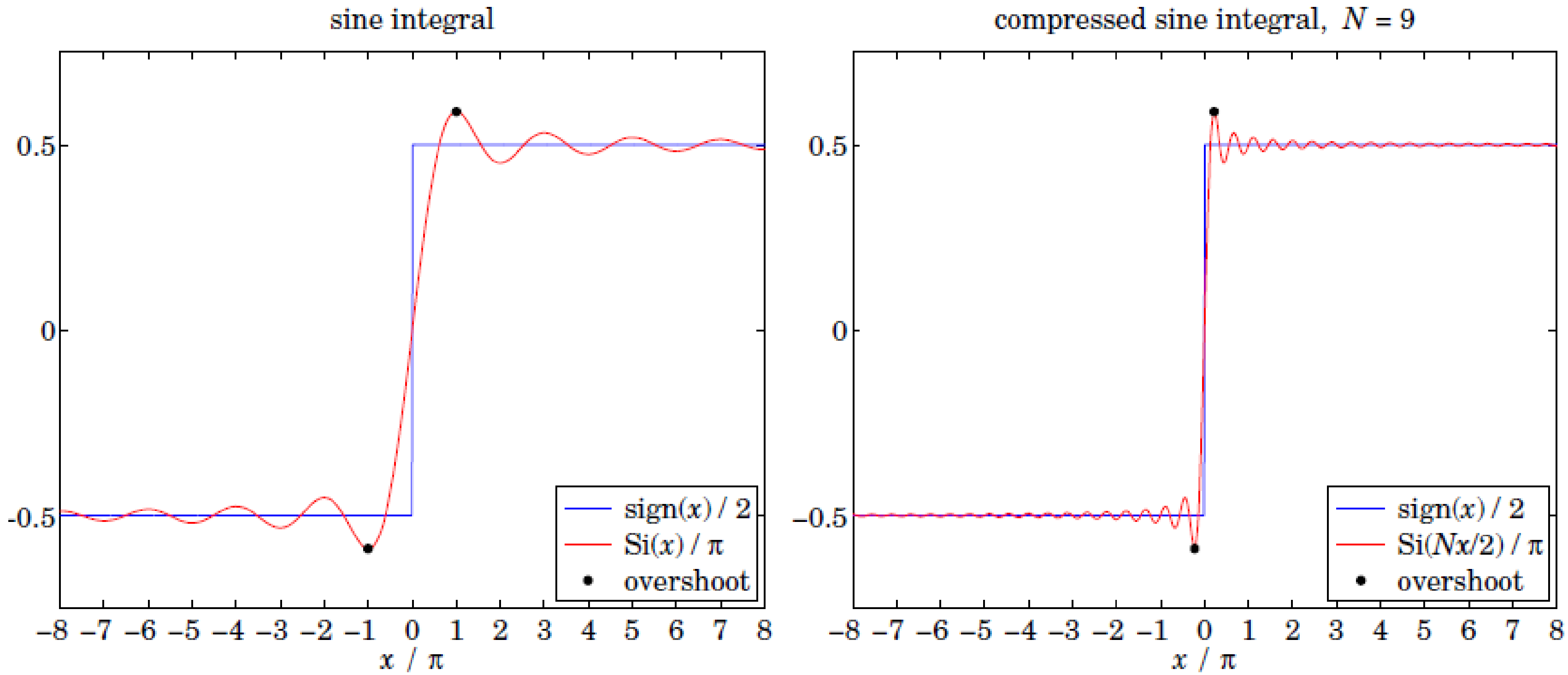
the Si function can be computed by the built-in function, **sinint**, or, by the function, **Si(x)**, on Sakai Resources.

$$\widehat{D}(\omega) = \frac{1}{\pi}\,\mathrm{Si}\big(N(\omega+\omega_c)/2\big) - \frac{1}{\pi}\,\mathrm{Si}\big(N(\omega-\omega_c)/2\big)$$

In fact, the two Si terms match the two sign terms of

$$D(\omega) = \frac{1}{2}\,\mathrm{sign}(\omega+\omega_c) - \frac{1}{2}\,\mathrm{sign}(\omega-\omega_c)$$

with the Gibbs overshoot arising from the properties of the Si function. The figure below demonstrates how the function $\mathrm{Si}(x)/\pi$ approximates the function $\mathrm{sign}(x)/2$, and how $\mathrm{Si}(Nx/2)/\pi$ approximates $\mathrm{sign}(x)/2$ even better, being a compressed version of $\mathrm{Si}(x)/\pi$.



The maximum of the Si function occurs at,

$$\frac{d}{dx}\mathrm{Si}(x) = \frac{\sin x}{x} = 0 \quad \Rightarrow \quad x = \pi$$

resulting in the maximum value, $\mathrm{Si}(\pi)/\pi = 0.58949$, shown on the left graph of the figure, whose deviation from the maximum value of the function $\mathrm{sign}(x)/2$, that is, from $1/2$, is the overshoot,

$$\boxed{\text{overshoot} = \frac{1}{\pi}\,\mathrm{Si}(\pi) - \frac{1}{2} = 0.58949 - 0.5 = 0.08949}$$

The maximum shown on right graph occurs at $Nx/2 = \pi$, or, $x = 2\pi/N$, and the maximum value is still the same, $\mathrm{Si}(\pi)/\pi = 0.58949$.

The Gibbs phenomenon is a peculiar property of Fourier series and it always appears in periodic waveforms that that have discontinuities. Replacing the rectangular window with a tapered non-rectangular one tends to diminish the overshoot. See the historical articles and reviews on Sakai Resources, as well as the Fourier series practice problem set, **s21set9.pdf**.

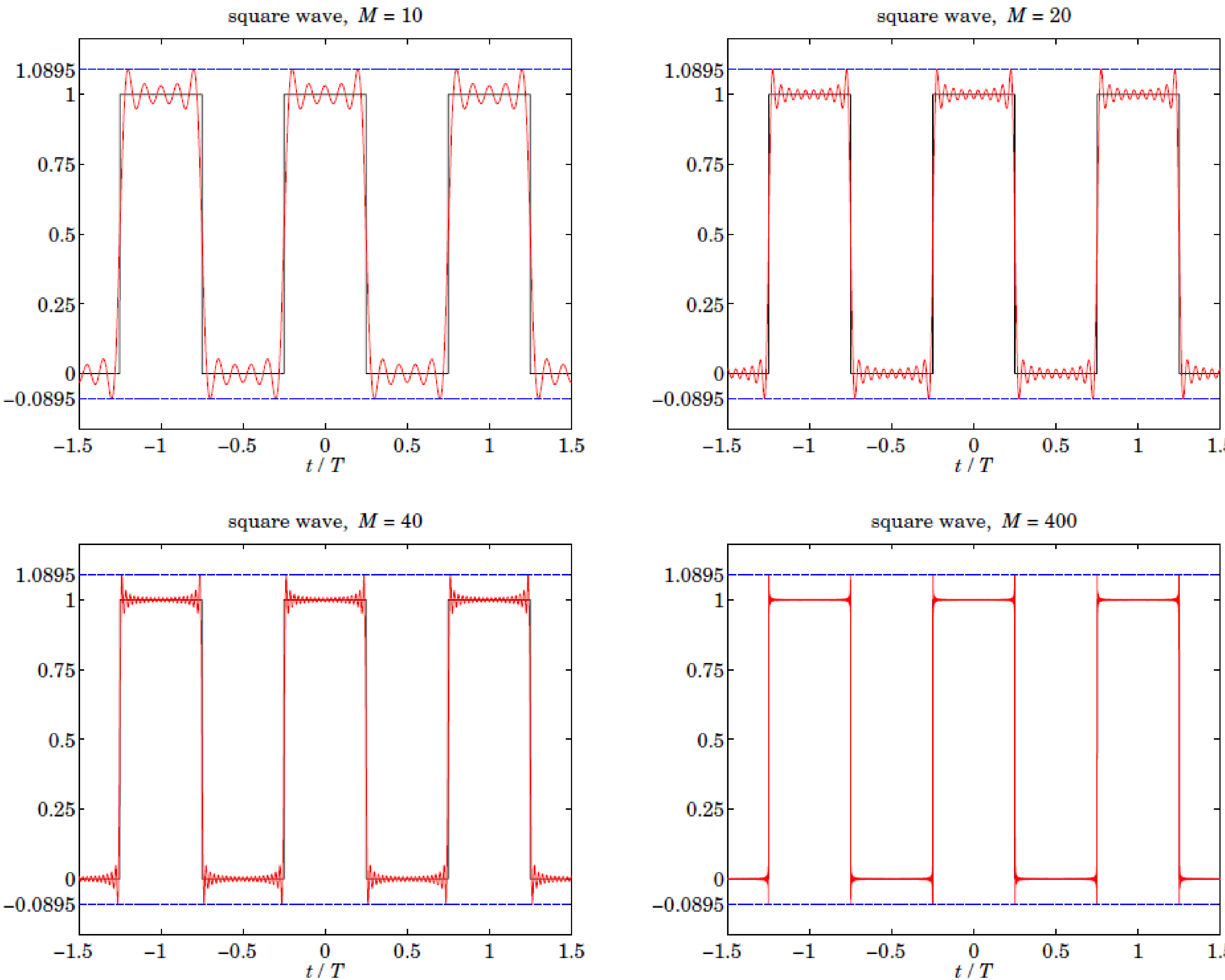


Fig. 1 Gibbs phenomenon for square wave – rectangular window.

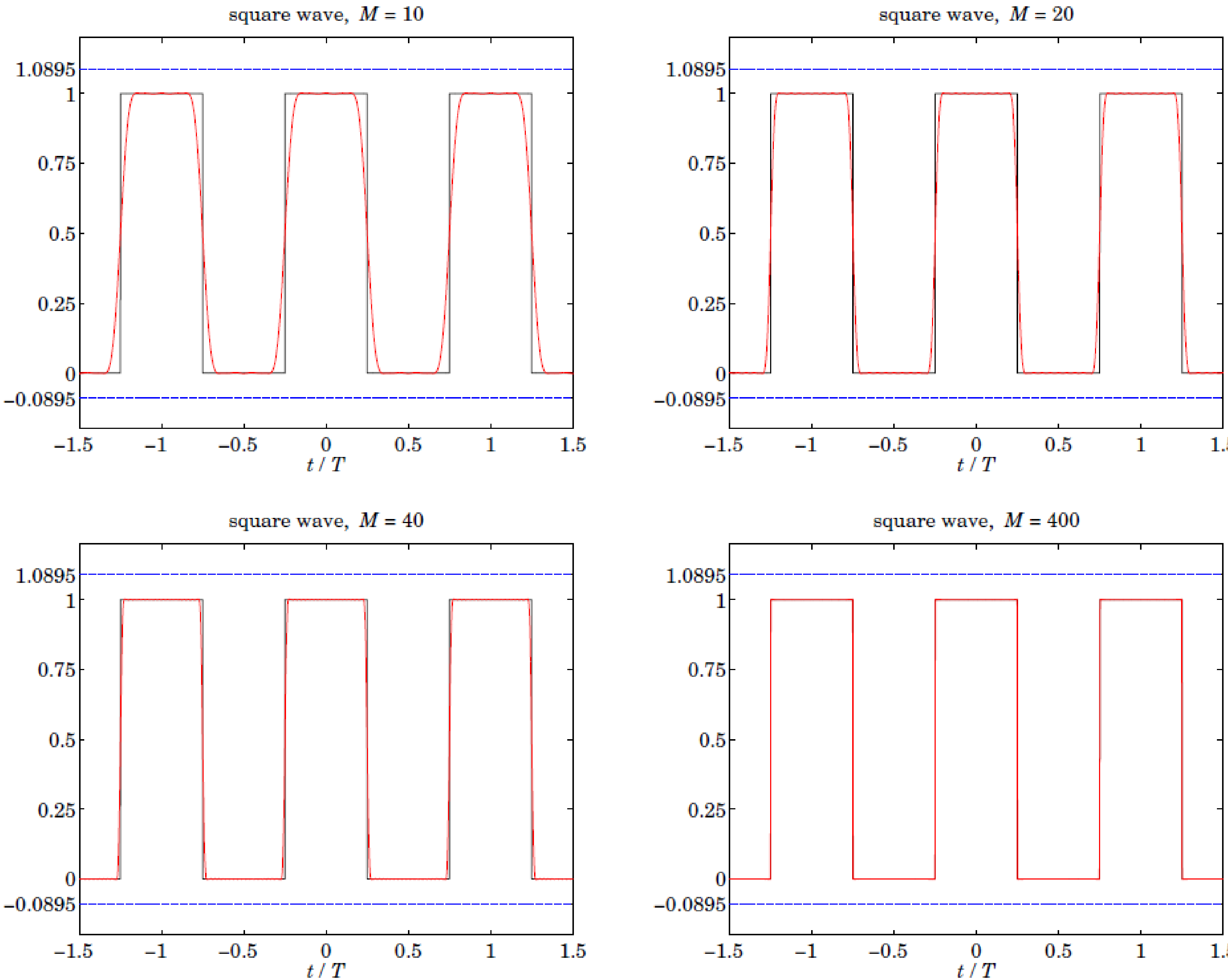


Fig. 2 Gibbs phenomenon for square wave – Hamming window.

Hamming Window

To eliminate the 8.9% passband and stopband ripples, we may replace the rectangular window $w(n)$ by a non-rectangular one, which tapers off gradually at its endpoints, thus reducing the ripple effect. The Hamming window is defined by,

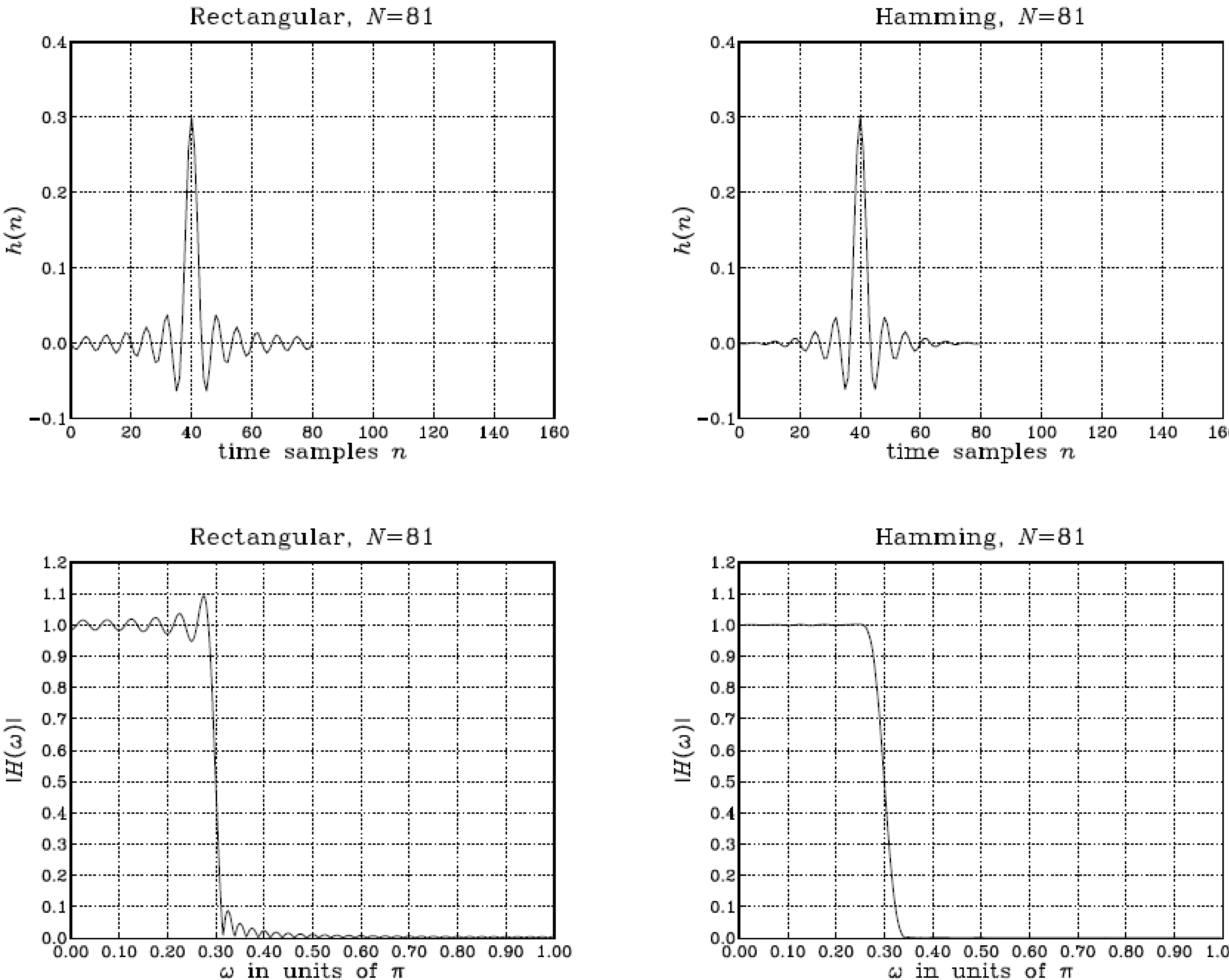
$$\boxed{w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \qquad n = 0, 1, \ldots, N-1}$$

In particular, the Hamming windowed impulse response for a length-N lowpass filter will be, where $N = 2M+1$ and $n = 0, 1, \ldots, N-1$:

$$h(n) = w(n)d(n-M) = \left[0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right] \cdot \frac{\sin\big(\omega_c(n-M)\big)}{\pi(n-M)}$$

As an example, consider the design of a length $N = 81$ lowpass filter with cutoff frequency $\omega_c = 0.3\pi$. The figure below shows the rectangularly and Hamming windowed impulse responses and the frequency responses. Note how the Hamming impulse response tapers off to zero more gradually.

The passband/stopband ripples of the rectangular window design are virtually eliminated from the Hamming window design. Actually, there are small ripples with maximum overshoot of about 0.2%, but they are not visible in the scale of the figure. The price for eliminating the ripples is loss of resolution, which is reflected into a wider transition width.

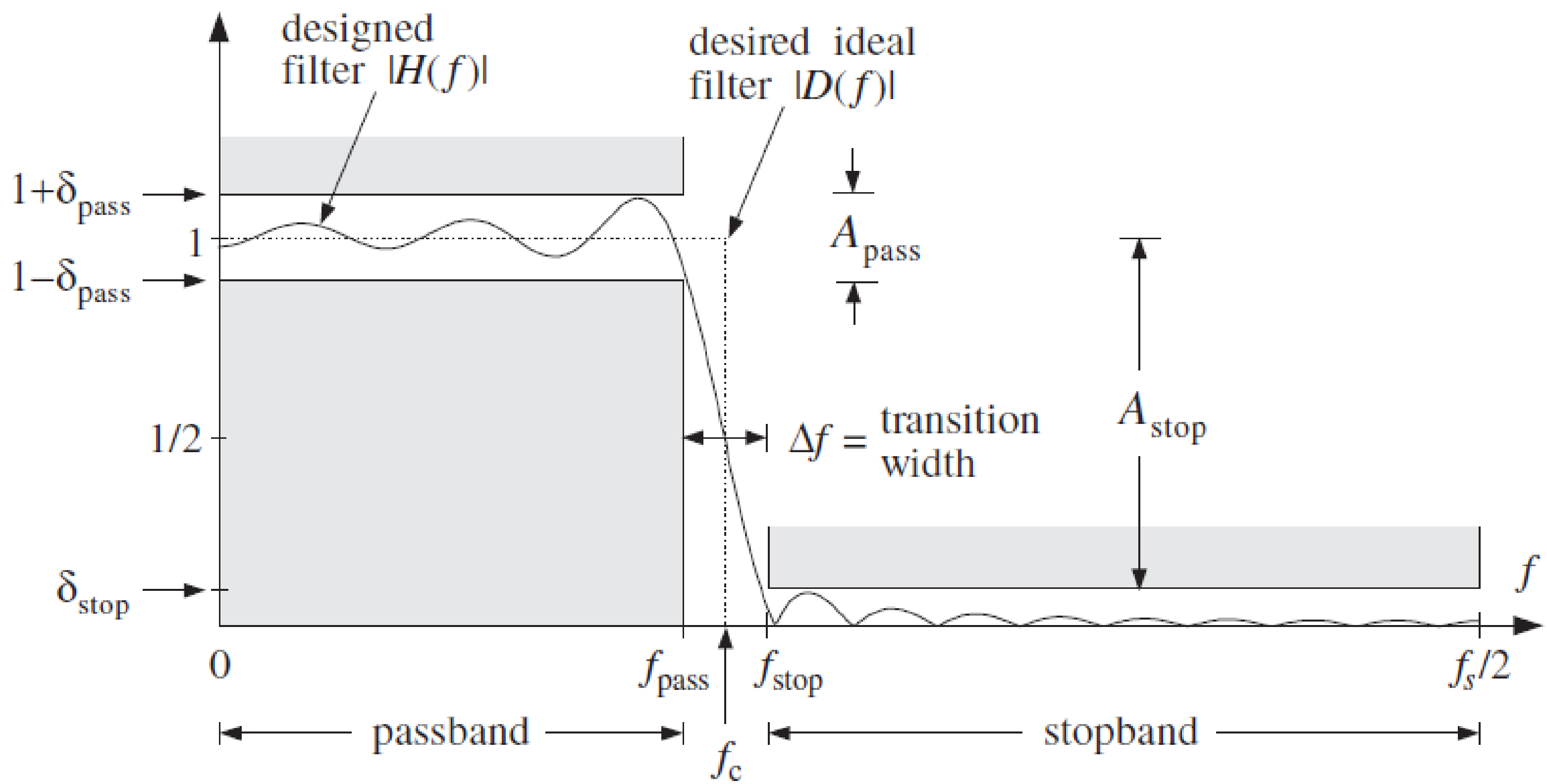


Rectangular, N=81
h(n)
time samples n
Hamming, N=81
h(n)
time samples n
Rectangular, N=81
|H(ω)|
ω in units of π
Hamming, N=81
|H(ω)|
ω in units of π

Kaiser Window for Filter Design

The rectangular and Hamming window designs are very simple, but do not provide good control over the filter design specifications. With these windows, the amount of overshoot is always fixed to 8.9% or 0.2% and cannot be changed to a smaller value if so desired.

A flexible set of specifications is shown in below, in which the designer can arbitrarily specify the amount of passband and stopband overshoot δ_{pass}, δ_{stop}, as well as the transition width Δf.



The passband/stopband frequencies $\{f_{\text{pass}}, f_{\text{stop}}\}$ are related to the ideal cut-off frequency f_c and transition width Δf by

$$f_c = \frac{1}{2}(f_{\text{pass}} + f_{\text{stop}}), \qquad \Delta f = f_{\text{stop}} - f_{\text{pass}}$$

Thus, f_c is chosen to lie exactly in the middle between f_{pass} and f_{stop}. We can also write,

$$f_{\text{pass}} = f_c - \frac{1}{2}\Delta f, \qquad f_{\text{stop}} = f_c + \frac{1}{2}\Delta f$$

The normalized versions of the frequencies are the digital frequencies:

$$\omega_{\text{pass}} = \frac{2\pi f_{\text{pass}}}{f_s}, \quad \omega_{\text{stop}} = \frac{2\pi f_{\text{stop}}}{f_s}, \quad \omega_c = \frac{2\pi f_c}{f_s}, \quad \Delta\omega = \frac{2\pi \Delta f}{f_s}$$

In practice, the passband/stopband overshoots are usually expressed in dB:

$$A_{\text{pass}} = 20 \log_{10}\left(\frac{1+\delta_{\text{pass}}}{1-\delta_{\text{pass}}}\right), \qquad A_{\text{stop}} = -20 \log_{10} \delta_{\text{stop}}$$

A simplified version of the passband equation can be obtained by expanding it to first order in δ_{pass}, giving:

$$A_{\text{pass}} \approx \frac{40}{\ln 10}\,\delta_{\text{pass}} = 17.3718\,\delta_{\text{pass}}$$

which is valid for small values of δ_{pass}. Inverting, we also have,

$$\delta_{\text{pass}} = \frac{10^{A_{\text{pass}}/20} - 1}{10^{A_{\text{pass}}/20} + 1} \approx \frac{A_{\text{pass}}}{17.3718}\,, \qquad \delta_{\text{stop}} = 10^{-A_{\text{stop}}/20}$$

Thus, one can pass back and forth between the specification sets:

$$\{f_{\text{pass}}, f_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\} \quad \Leftrightarrow \quad \{f_c, \Delta f, \delta_{\text{pass}}, \delta_{\text{stop}}\}$$

Although δ_{pass} and δ_{stop} can be specified independently of each other, it is a property of *all* window designs that the final designed filter will have *equal* passband and stopband ripples. Therefore, we must design the filter on the basis of the *smaller* of the two ripples, that is,

$$\boxed{\delta = \min(\delta_{\text{pass}}, \delta_{\text{stop}})}$$

The designed filter will have passband and stopband ripple equal to δ. The value of δ can also be expressed in dB:

$$\boxed{A = -20\log_{10}\delta\,, \qquad \delta = 10^{-A/20}}$$

In practice, the design is usually based on the stopband ripple δ_{stop}. This is so because any reasonably good choices for the passband and stopband attenuations (e.g., $A_{\text{pass}} = 0.1$ dB and $A_{\text{stop}} = 60$ dB) will almost always result into $\delta_{\text{stop}} < \delta_{\text{pass}}$, and therefore, $\delta = \delta_{\text{stop}}$, and in dB, $A = A_{\text{stop}}$. Thus, it is useful to think of A as the stopband attenuation.

The main limitation of most windows is that they have a *fixed* value of δ, which depends on the particular window shape. Such windows limit the achievable passband and stopband attenuations $\{A_{\text{pass}}, A_{\text{stop}}\}$ to only certain specific values.

For example, the following table shows the attenuations achievable by the rectangular and Hamming windows, with the values $\delta = \delta_{\text{pass}} = \delta_{\text{stop}} = 0.089$ and $\delta = \delta_{\text{pass}} = \delta_{\text{stop}} = 0.002$, respectively. The table also shows the corresponding value of the transition width parameter D, defined below.

Window	δ	A_{stop} dB	A_{pass} dB	D
Rectangular	8.9%	-21	1.55	0.92
Hamming	0.2%	-54	0.03	3.21
Kaiser	variable δ	$-20\log_{10}\delta$	$17.3718\,\delta$	$(A-7.95)/14.36$

The only windows that do not suffer from the above limitation are the Kaiser window, the Dolph-Chebyshev window, and the Saramäki windows (see I2SP references). These windows have an adjustable shape parameter that allows the window to achieve any desired value of ripple δ or attenuation A.

The Kaiser window is unique in the above class in that it has near-optimum performance (in the sense of minimizing the sidelobe energy of the window), as well as having the simplest implementation. It depends on two parameters: its length N and the shape parameter α. Assuming odd length $N = 2M + 1$, the window is defined as in the spectral analysis context, as follows, for $n = 0, 1, \ldots, N - 1$,

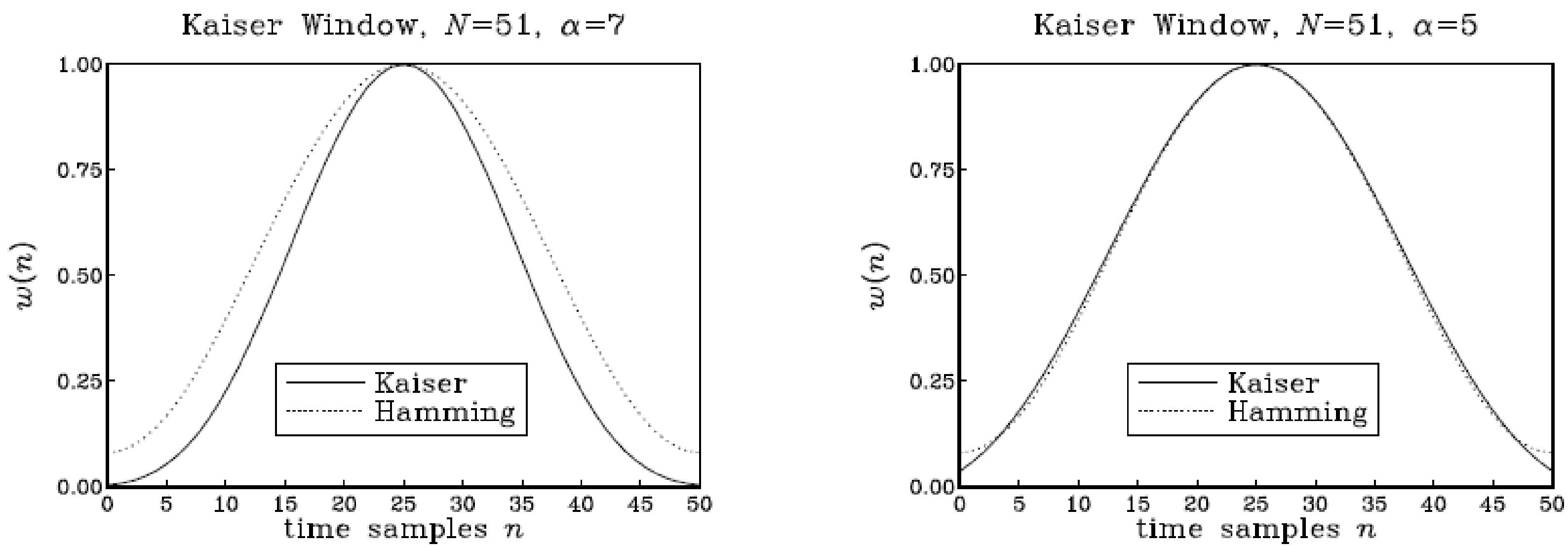
$$\text{(Kaiser window)} \qquad w(n) = \frac{I_0\left(\alpha\sqrt{1 - (n - M)^2/M^2}\right)}{I_0(\alpha)}$$

The numerator can be rewritten in the following form, which is more convenient for numerical evaluation:

$$w(n) = \frac{I_0\big(\alpha\sqrt{n(2M - n)}/M\big)}{I_0(\alpha)}, \quad n = 0, 1, \ldots, N - 1$$

Like all window functions, the Kaiser window is *symmetric* about its middle, $n = M$, and has the value $w(M) = 1$ there. At the endpoints, $n = 0$ and $n = N - 1$, it has the value $1/I_0(\alpha)$ because $I_0(0) = 1$.

The figure below compares a Hamming window of length $N = 51$ to the Kaiser windows of the same length and shape parameters $\alpha = 7$ and $\alpha = 5$. For $\alpha = 5$ the Kaiser and Hamming windows agree closely, except near their endpoints. For $\alpha = 0$ the Kaiser window reduces to the rectangular one.



The window parameters $\{N, \alpha\}$ are computable in terms of the filter specifications, namely, the ripple δ and transition width Δf, by the following design equations developed by Kaiser. The parameter α is calculated from:

$$\alpha = \begin{cases} 0.1102(A-8.7), & \text{if } A \geq 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & \text{if } 21 < A < 50 \\ 0, & \text{if } A \leq 21 \end{cases}$$

The filter length N is inversely related to the transition width:

$$\Delta f = D\,\frac{f_s}{N-1} \quad \Leftrightarrow \quad N-1 = \frac{D\,f_s}{\Delta f}$$

where the factor D is computed also in terms of A by

$$D = \begin{cases} \dfrac{A-7.95}{14.36}, & \text{if } A > 21 \\ 0.922, & \text{if } A \leq 21 \end{cases}$$

For the practical range of $A \geq 50$ dB, the formulas simplify into,

$$\alpha = 0.1102(A-8.7)\,, \qquad D = \frac{A-7.95}{14.36} \qquad (\text{for } A \geq 50 \text{ dB})$$

To summarize, the steps for designing a lowpass filter are as follows. Given the specifications $\{f_{\text{pass}}, f_{\text{stop}}, A_{\text{pass}}, A_{\text{stop}}\}$:

1. Calculate f_c and Δf. Then, calculate $\omega_c = 2\pi f_c/f_s$.
2. Calculate δ_{pass} and δ_{stop}.
3. Calculate $\delta = \min(\delta_{\text{pass}}, \delta_{\text{stop}})$ and $A = -20\log_{10}\delta$ in dB.
4. Calculate α and D from the above design equations.
5. Calculate the filter length N and round it up to the next *odd* integer, $N = 2M + 1$, and set $M = (N-1)/2$.
6. Calculate the Kaiser window function, $w(n)$, $n = 0, 1, \ldots, N-1$.
7. Calculate the windowed impulse response, for $n = 0, 1, \ldots, N-1$,

$$h(n) = w(n)d(n-M) = w(n)\cdot\frac{\sin\big(\omega_c(n-M)\big)}{\pi(n-M)}$$

Note that the window parameters $\{N, \alpha\}$ depend only on the specifications $\{A, \Delta f\}$ and not on f_c. However, $h(n)$ does depend on f_c. The design steps can be modified easily to design highpass, bandpass and banstop filters. See I2SP-Ch.10 for the details.

the following functions from **I2SP-Appendix** implement the Kaiser window design of lowpass, highpass, bandspass, differentiator, and Hilbert transformer filters

```
kbp     - bandpass FIR filter design
kdiff   - lowpass FIR differentiator design
khilb   - lowpass FIR Hilbert trasformer design
klh     - lowpass/highpass FIR filter design
kparm   - Kaiser window parameters for filter design
kparm2  - Kaiser window parameters for spectral analysis
kwind   - Kaiser window
```

Example:
Using the Kaiser window, design a lowpass digital filter with the following specifications:

$$\begin{aligned} &f_s = 20 \text{ kHz} \\ &f_{\text{pass}} = 4 \text{ kHz}, \quad f_{\text{stop}} = 5 \text{ kHz} \\ &A_{\text{pass}} = 0.1 \text{ dB}, \quad A_{\text{stop}} = 80 \text{ dB} \end{aligned}$$

Solution:
First, we calculate δ_{pass} and δ_{stop},

$$\delta_{\text{pass}} = \frac{10^{0.1/20} - 1}{10^{0.1/20} + 1} = 0.0058, \qquad \delta_{\text{stop}} = 10^{-80/20} = 0.0001$$

Therefore, $\delta = \min(\delta_{\text{pass}}, \delta_{\text{stop}}) = \delta_{\text{stop}} = 0.0001$, which in dB is $A = -20\log_{10}\delta = A_{\text{stop}} = 80$. The D and α parameters are computed by:

$$\alpha = 0.1102(A-8.7) = 0.1102(80-8.7) = 7.857, \quad D = \frac{A - 7.95}{14.36} = 5.017$$

The filter width and ideal cutoff frequency are:

$$\Delta f = f_{\text{stop}} - f_{\text{pass}} = 1 \text{ kHz}, \quad f_c = \frac{1}{2}(f_{\text{pass}} + f_{\text{stop}}) = 4.5 \text{ kHz}$$

$$\omega_c = \frac{2\pi f_c}{f_s} = 0.45\pi$$

The filter length (rounded up to the nearest odd integer) is,

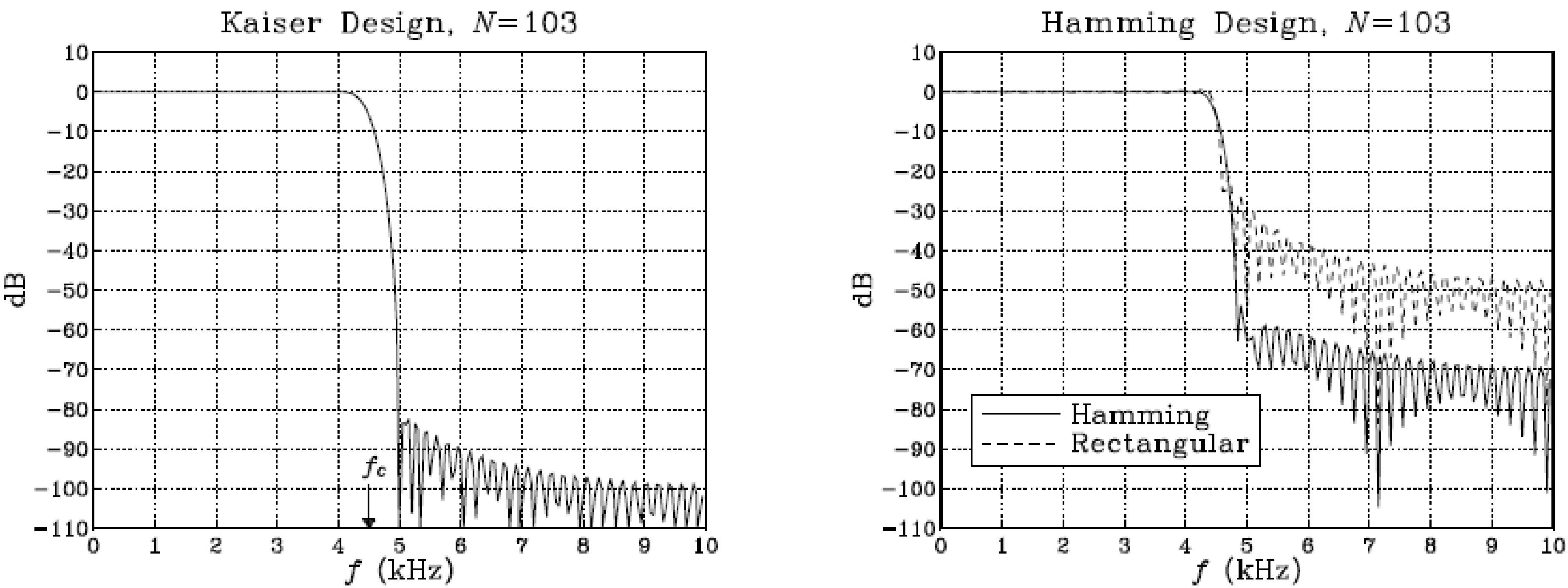
$$N = 1 + \frac{Df_s}{\Delta f} = 101.35 \quad \Rightarrow \quad N = 103, \quad M = \frac{1}{2}(N-1) = 51$$

The windowed impulse response will be, for $n = 0, 1, \ldots, 102$:

$$h(n) = w(n)d(n-M) = \frac{I_0\big(7.857\sqrt{n(102-n)}/51\big)}{I_0(7.857)} \cdot \frac{\sin\big(0.45\pi(n-51)\big)}{\pi(n-51)}$$

with $h(51) = \omega_c/\pi = 0.45$. The figure below shows the magnitude response in dB of $h(n)$, that is, $20\log_{10}|H(\omega)|$.

The transition width is extending from 4 to 5 kHz and the stopband specification is defined by the horizontal grid line at -80 dB. The passband specification is more than satisfied. It is $A_{\text{pass}} \simeq 17.3718\delta = 0.0017$ dB.



The figure also shows the corresponding Hamming and rectangularly windowed designs for the same length of $N = 103$. They both have a smaller transition width—the rectangular one even more so, but their stopband attenuations are limited to the standard values of 54 dB and 21 dB, respectively.

Frequency Sampling Method

The window method is very convenient for designing ideally shaped filters, primarily because the inverse DTFT frequency integral can be carried out in closed form.

For arbitrarily shaped frequency responses $D(\omega)$, we may use the frequency sampling method, in which the inverse DTFT integral is replaced by the approximate sum:

$$\boxed{\widetilde{d}(k) = \frac{1}{N} \sum_{i=-M}^{M} D(\omega_i) e^{j\omega_i k}, \qquad -M \leq k \leq M}$$

where $N = 2M + 1$. The approximation is essentially an inverse N-point DFT, with the DFT frequencies ω_i spanning equally the interval $[-\pi, \pi]$, instead of the standard DFT interval $[0, 2\pi]$:

$$\omega_i = \frac{2\pi i}{N}, \qquad -M \leq i \leq M$$

The rest of the window method may be applied as before, that is, given an appropriate length-N window $w(n)$, the final designed filter will be the delayed and windowed version of $\widetilde{d}(k)$:

$$h(n) = w(n)\widetilde{d}(n-M), \qquad n = 0, 1, \dots, N-1$$

We will discuss some examples of the frequency sampling method in I2SP-Ch.12, where we will design FIR filters for equalizing the slight passband droop of D/A converters and imperfect analog anti-image postfilters.

Other FIR Design Methods

The Kaiser window method is simple and flexible and can be applied to a variety of filter design problems. However, it does not always result in the smallest possible filter length N, which may be required in some very stringent applications.

The Parks-McClellan method based on the so-called optimum *equiripple Chebyshev approximation* generally results in shorter filters. Kaiser has shown that the filter length can be estimated in such cases by the following expression that uses the *geometric mean* of the two ripples, $\delta_g = \sqrt{\delta_{\text{pass}}\delta_{\text{stop}}}$,

$$N - 1 = \frac{Df_s}{\Delta f}, \qquad D = \frac{A_g - 13}{14.6}, \quad A_g = -20\log_{10}(\delta_g)$$