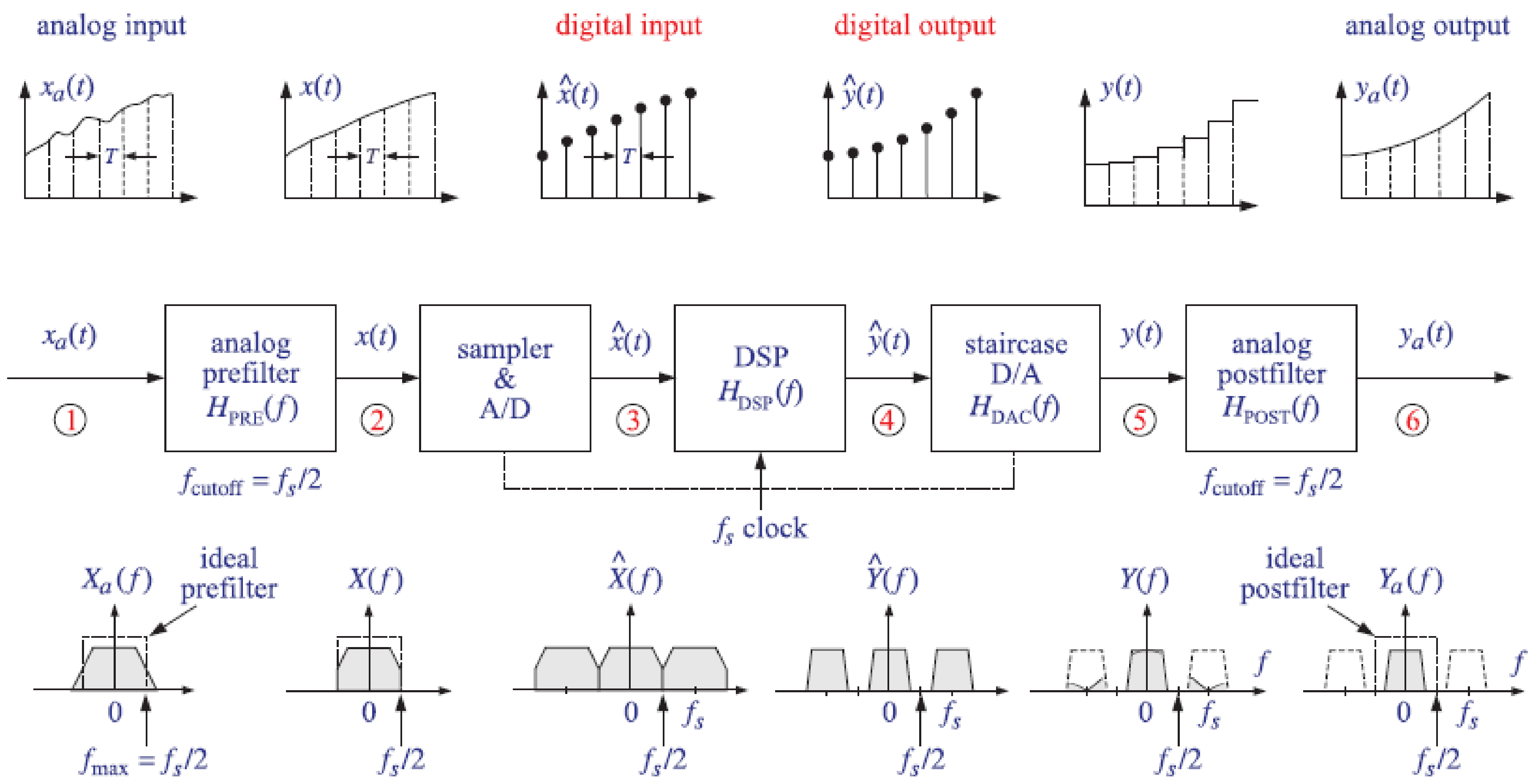
DSA – Feb. 1, 2021

Topics: Sampling, reconstruction, anti-aliasing prefilters, anti-image postfilters, staircase reconstructors and equalization, bandlimited functions, LTI systems, impulse response, frequency response, sinusoidal response.



Sampling Theorem

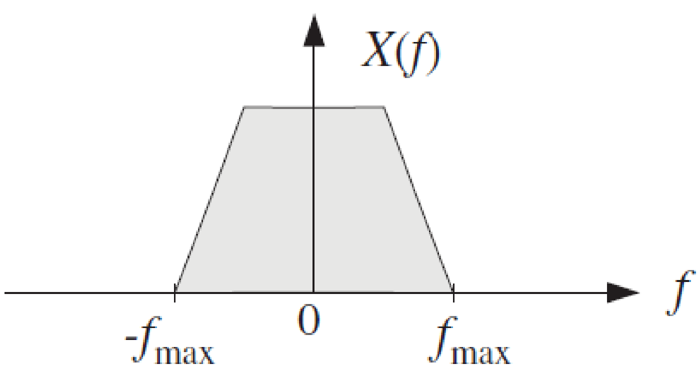
1. The signal $x(t)$ must be *bandlimited*, that is, its frequency spectrum must be limited to contain frequencies up to some maximum frequency, say $f_{\max}$, and no frequencies beyond that. A typical bandlimited spectrum is shown in Fig. 1.3.4.

2. The sampling rate f_s must be chosen to be at least *twice* the maximum frequency $f_{\max}$, that is,

$$\boxed{f_s \geq 2f_{\max}} \tag{1.3.2}$$

   or, expressed in terms of the sampling time interval,

$$\boxed{T \leq \frac{1}{2f_{\max}}}$$

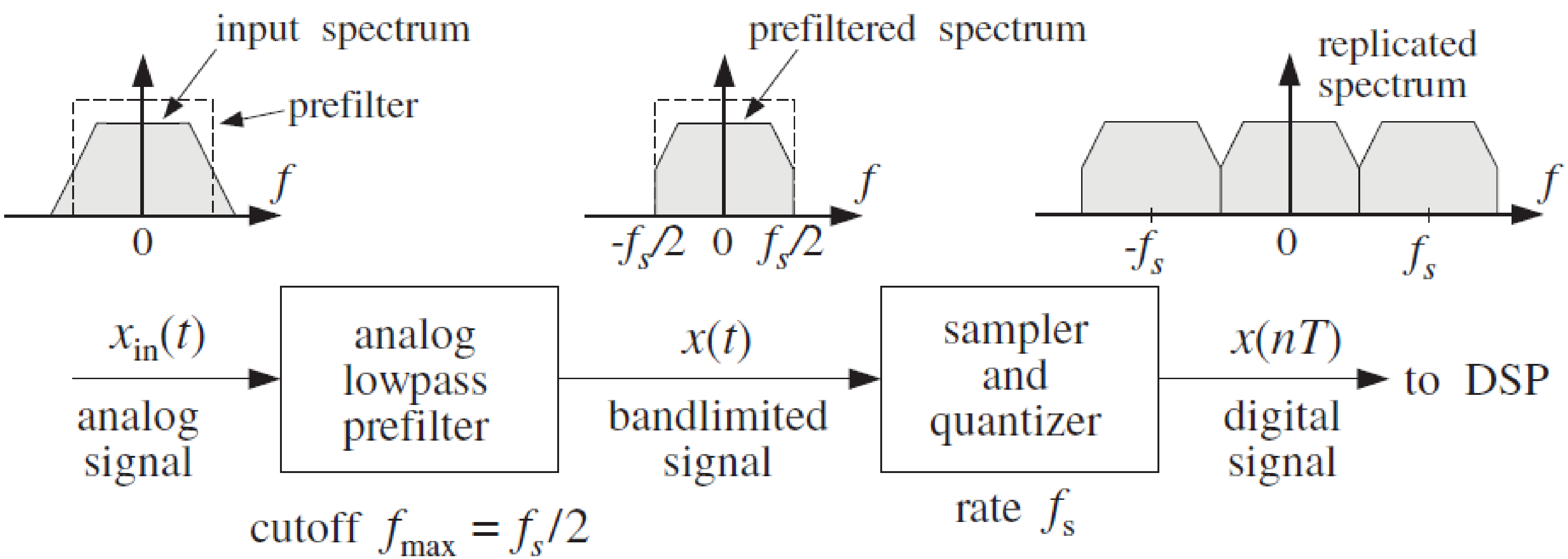


$$(f_s)_{\max} = 2f_{\max} = \text{Nyquist rate}$$

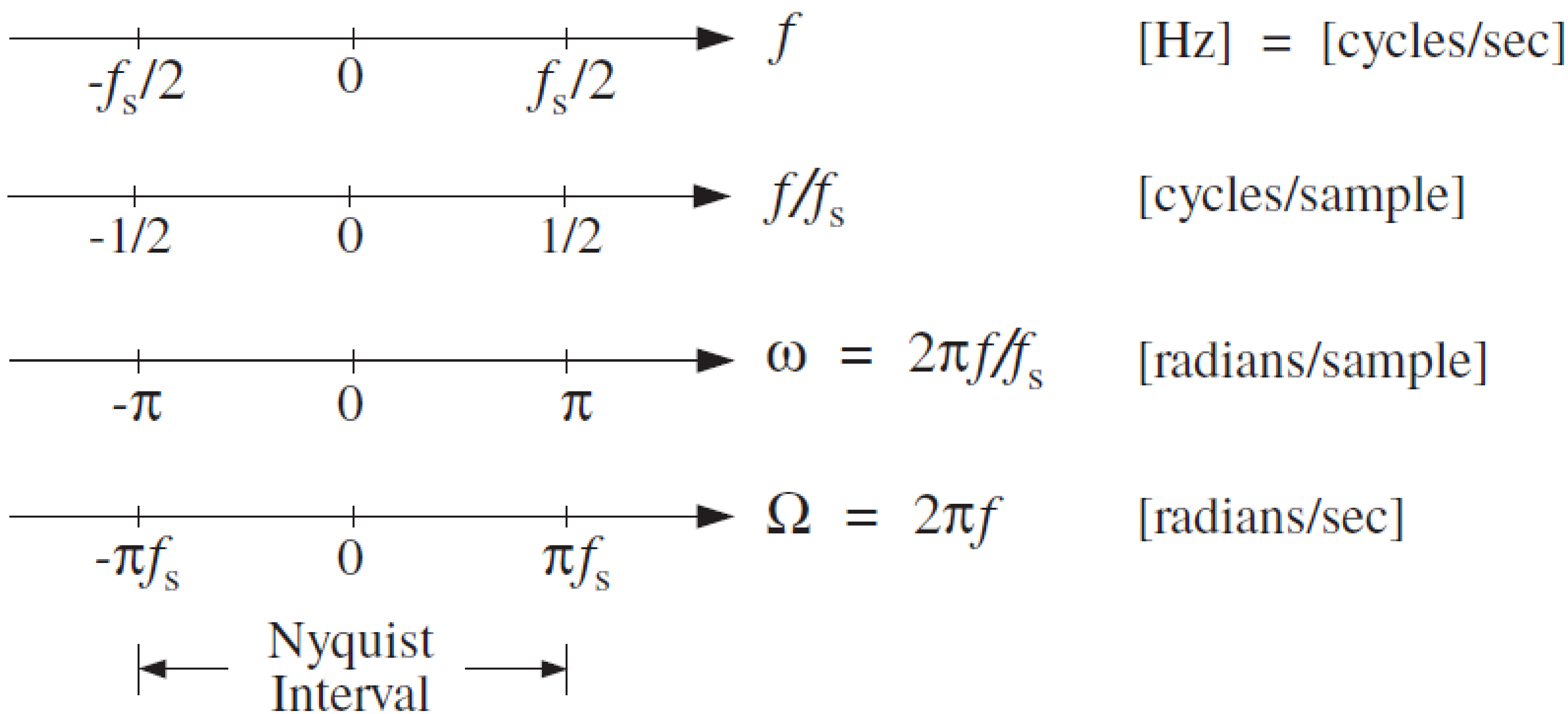
$$\frac{f_s}{2} = \text{Nyquist frequency}$$

$$\left[-\frac{f_s}{2}, \frac{f_s}{2}\right] = \text{Nyquist Interval}$$

Antialiasing Prefilters



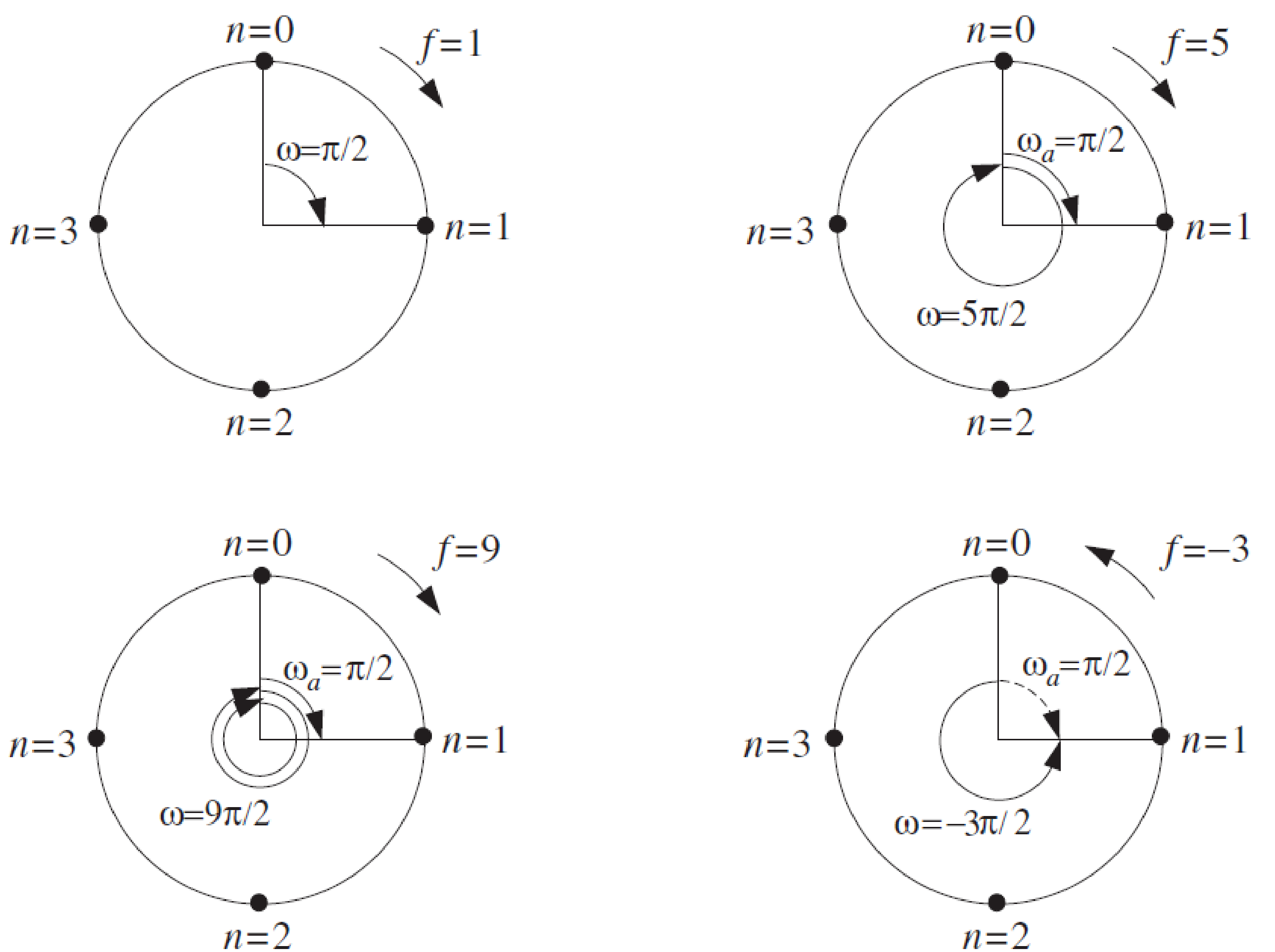
Common Frequency Units



Aliasing in Rotational Motion

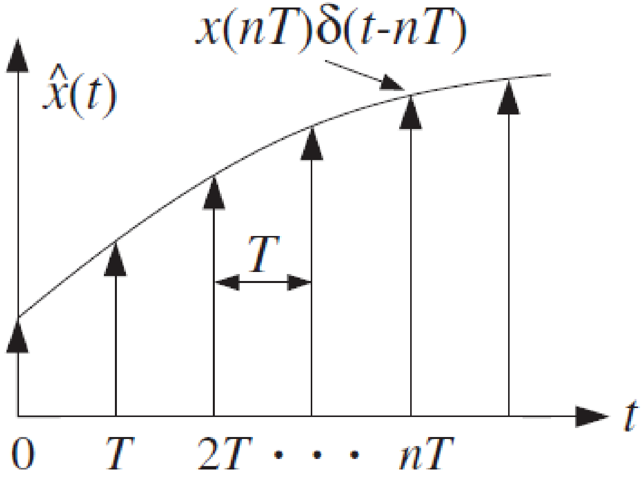
$$\omega = \Omega T = 2\pi f T = \frac{2\pi f}{f_s}$$

Example 1.4.10: Consider two wheels turning clockwise, one at $f_1 = 1$ Hz and the other at $f_2 = 5$ Hz, as shown below. Both are sampled with a strobe light flashing at $f_s = 4$ Hz. Note that the second one is turning at $f_2 = f_1 + f_s$.



Spectra of Sampled Signals

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-2\pi j f t}\, dt$$



$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

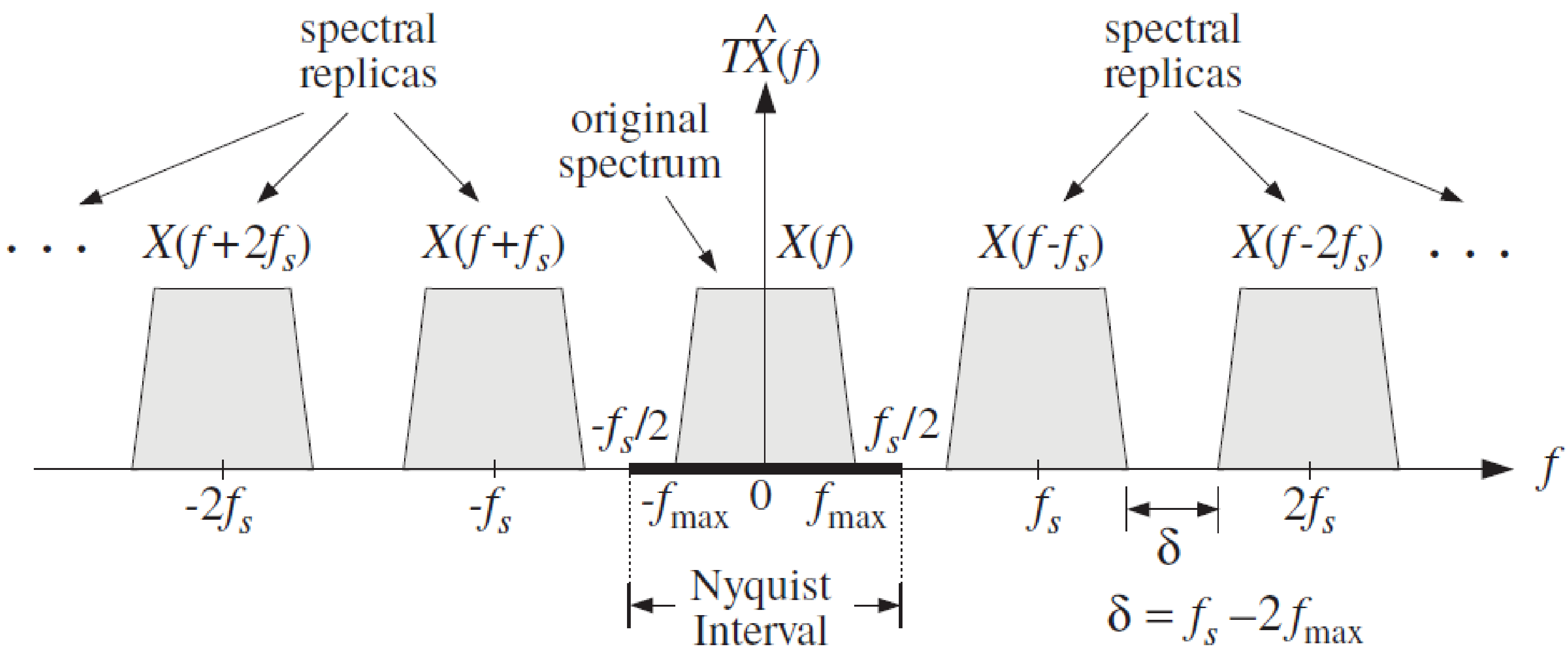
$$f_s = \frac{1}{T}$$

$$\hat{X}(f) = \int_{-\infty}^{\infty} \hat{x}(t)\, e^{-2\pi j f t}\, dt$$

$$\boxed{\hat{X}(f) = \sum_{n=-\infty}^{\infty} x(nT) e^{-2\pi j f n T} = \frac{1}{T} \sum_{m=-\infty}^{\infty} X(f - m f_s)}$$

Spectra of Sampled Signals

$$\hat{X}(f) = \sum_{n=-\infty}^{\infty} x(nT)e^{-2\pi j f n T} = \frac{1}{T}\sum_{m=-\infty}^{\infty} X(f - mf_s)$$



Spectra of Sampled Signals

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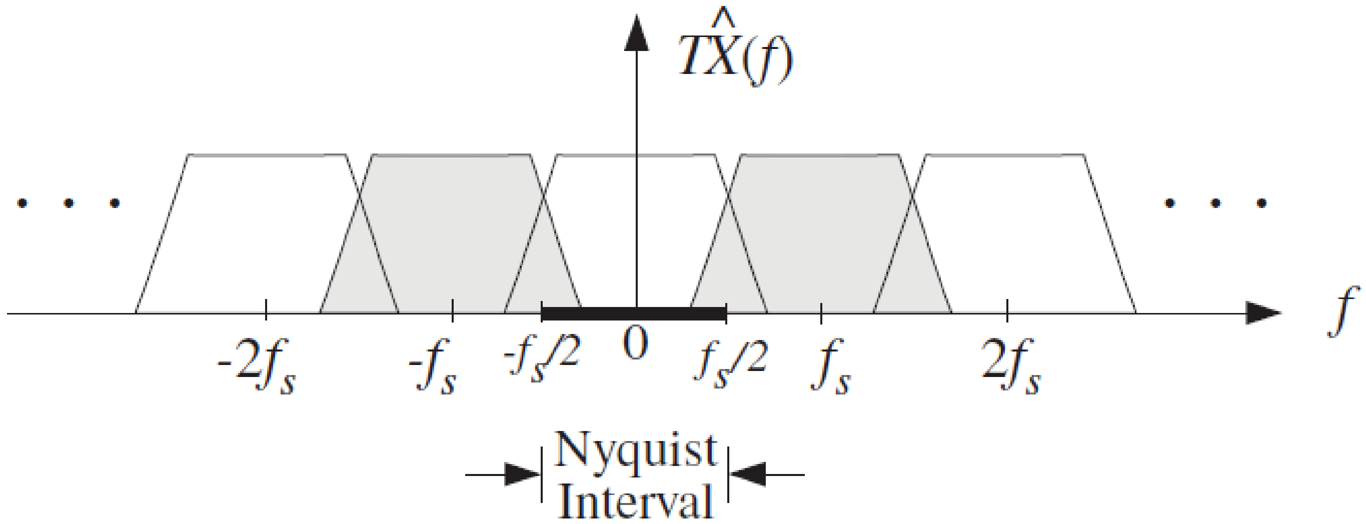


Spectra of Sampled Signals

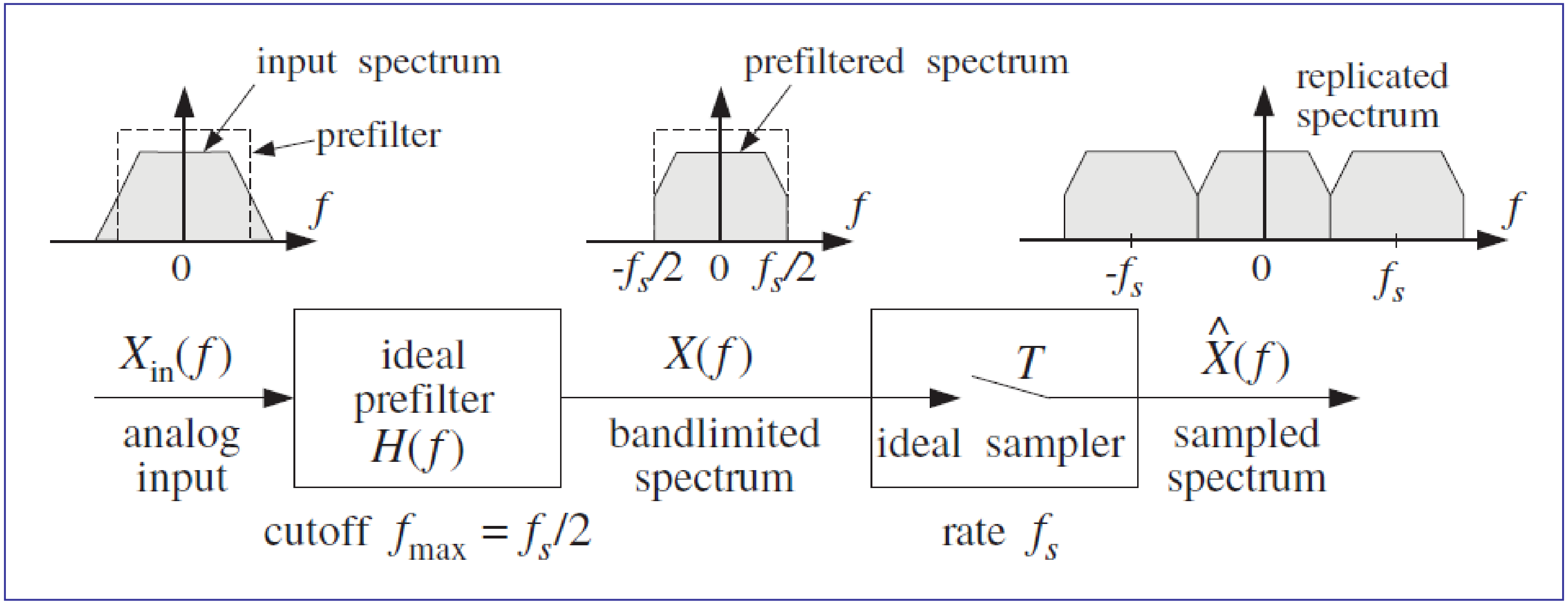
$$\hat{X}(f) = \sum_{n=-\infty}^{\infty} x(nT)e^{-2\pi j f n T} = \frac{1}{T}\sum_{m=-\infty}^{\infty} X(f - m f_s)$$

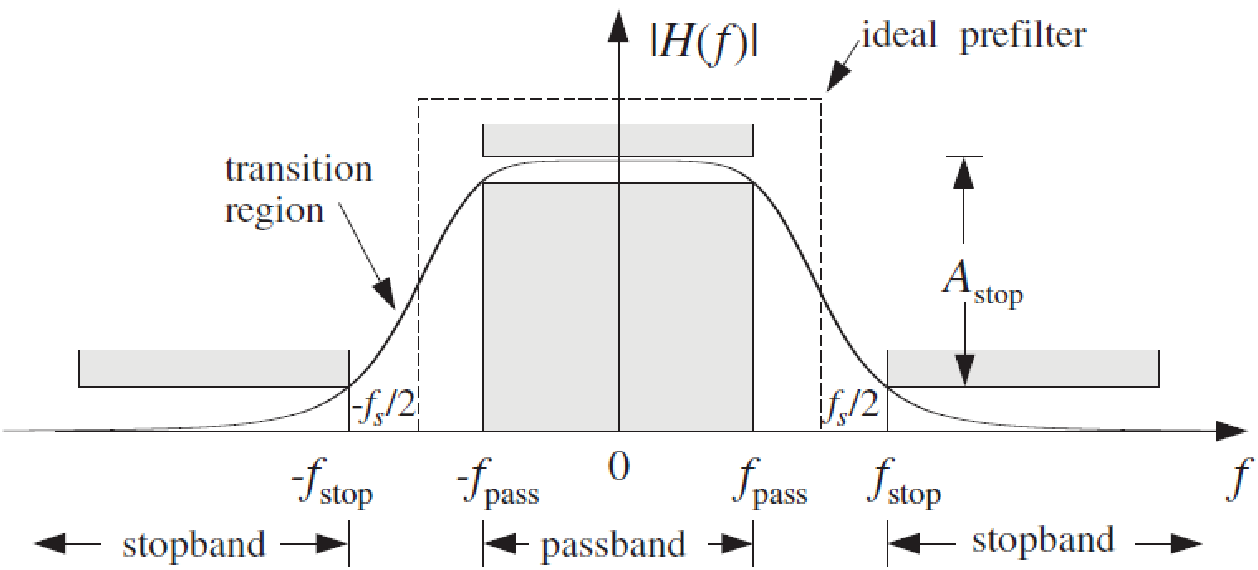
Discrete-Time Fourier Transform (DTFT)

$$\hat{X}(\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega n}$$

$$\omega = 2\pi f T = \frac{2\pi f}{f_s}$$

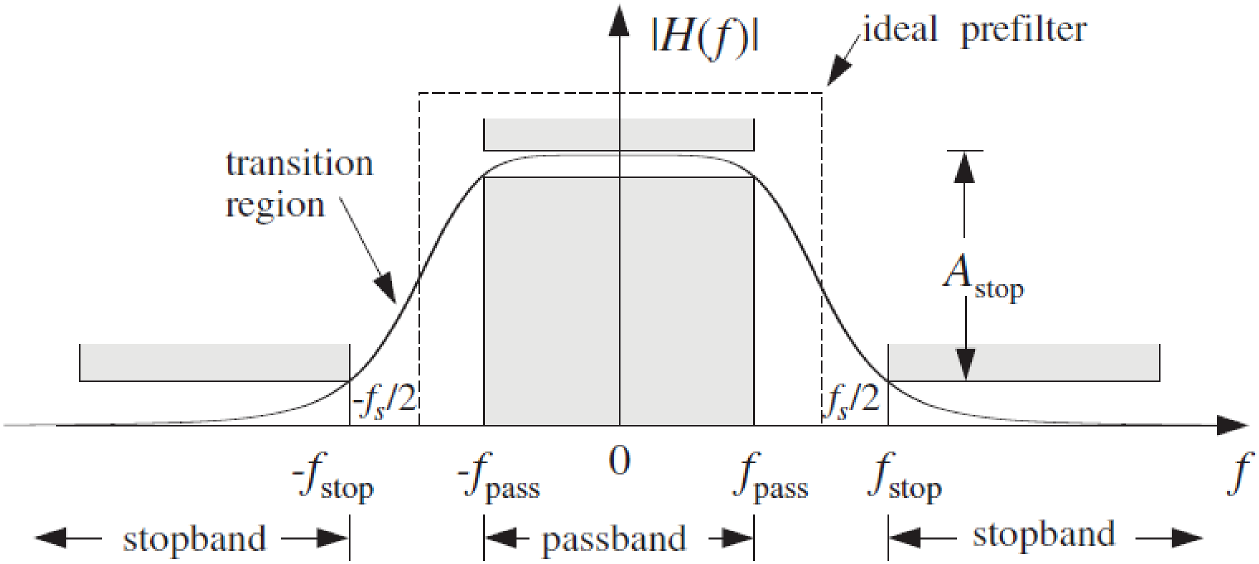
Practical Antialiasing Prefilters





$$f_{\text{pass}} = f_{\max}$$
$$f_{\text{stop}} = f_s - f_{\text{pass}}$$

Practical Antialiasing Prefilters



$$f_{\text{pass}} = f_{\text{max}}$$
$$f_{\text{stop}} = f_s - f_{\text{pass}}$$

attenuation in dB

$$A(f) = -20\,\log_{10}\left|\frac{H(f)}{H(f_{\text{ref}})}\right|$$

$$|H(f)| = |H(f_{\text{ref}})| \cdot 10^{-A(f)/20}$$

typical asymptotic behavior

$$A(f) \approx 6N\,\log_2\left(\frac{f}{f_{\text{ref}}}\right) = \text{dB/octave}$$

$$A(f) \approx 20N\,\log_{10}\left(\frac{f}{f_{\text{ref}}}\right) = \text{dB/decade}$$

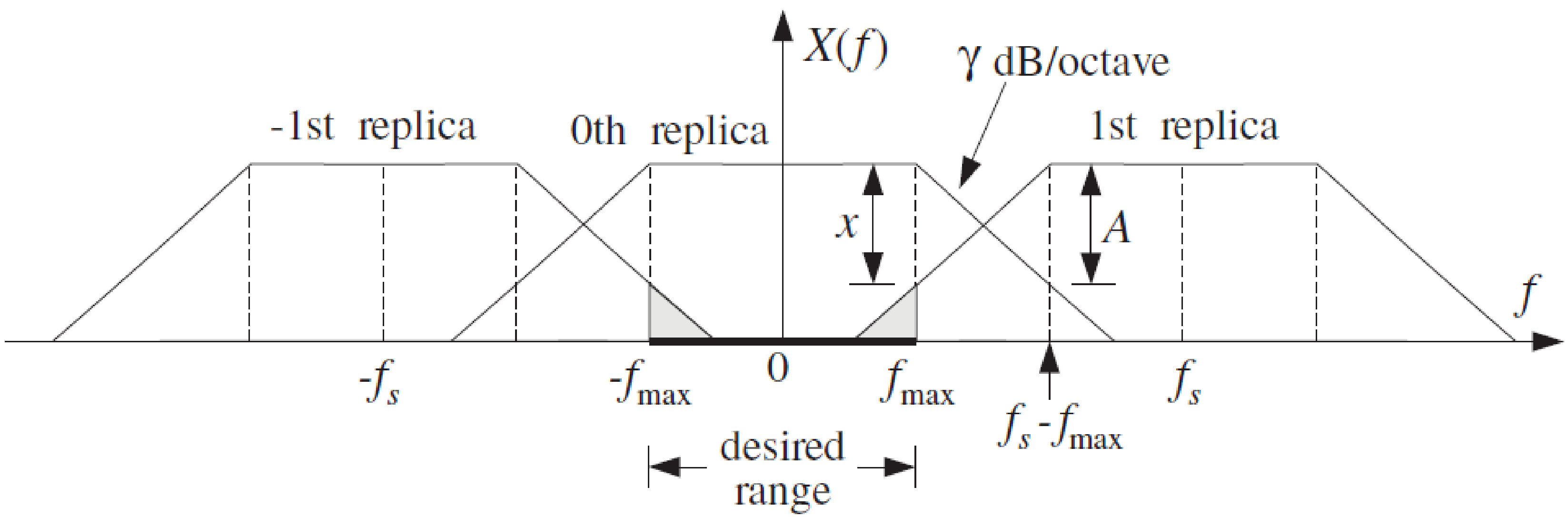
Practical Antialiasing Prefilters

Example 1.5.4: The significant frequency range of a signal extends to $f_{\max}$. Beyond $f_{\max}$, the spectrum attenuates by α dB/octave. We have available an off-the-shelf antialiasing prefilter that has a flat passband up to $f_{\max}$ and attenuates by β dB/octave beyond that. It is required that within the $f_{\max}$ range of interest, the aliased components due to sampling be suppressed by more than A dB. Show that the *minimum* sampling rate that we should use is given by

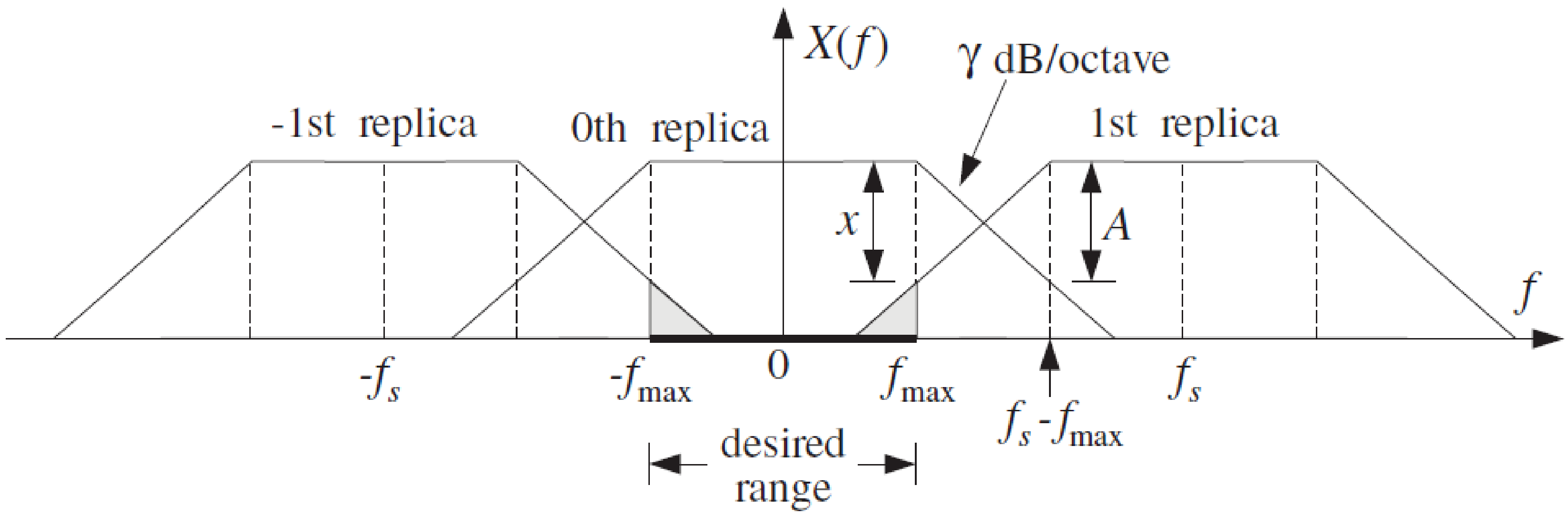
$$f_s = f_{\max} + 2^{A/\gamma} f_{\max}$$

where $\gamma = \alpha + \beta$.

$$\boxed{\begin{aligned} f_{\text{pass}} &= f_{\max} \\ f_{\text{stop}} &= f_s - f_{\text{pass}} \end{aligned}}$$



Practical Antialiasing Prefilters

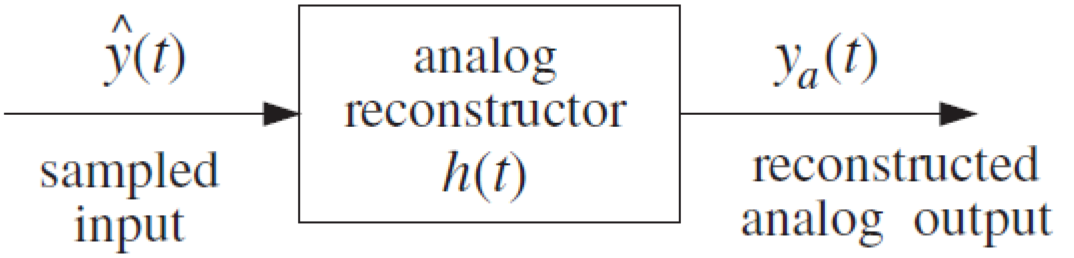


$$A_X(f) = \alpha \log_2\left(\frac{f}{f_{\max}}\right) + \beta \log_2\left(\frac{f}{f_{\max}}\right) = \gamma \log_2\left(\frac{f}{f_{\max}}\right)$$

$$A_X(f_s - f_{\max}) \geq A \qquad \Rightarrow \qquad \gamma \log_2\left(\frac{f_s - f_{\max}}{f_{\max}}\right) \geq A$$

$$\boxed{f_s = f_{\max} + 2^{A/\gamma} f_{\max}}$$

Analog Reconstructors



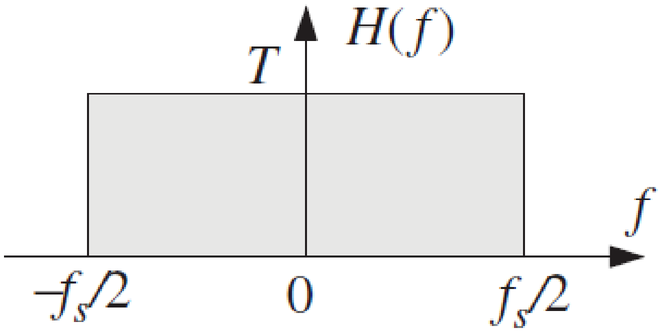
$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} y(nT)\,\delta(t-nT)$$

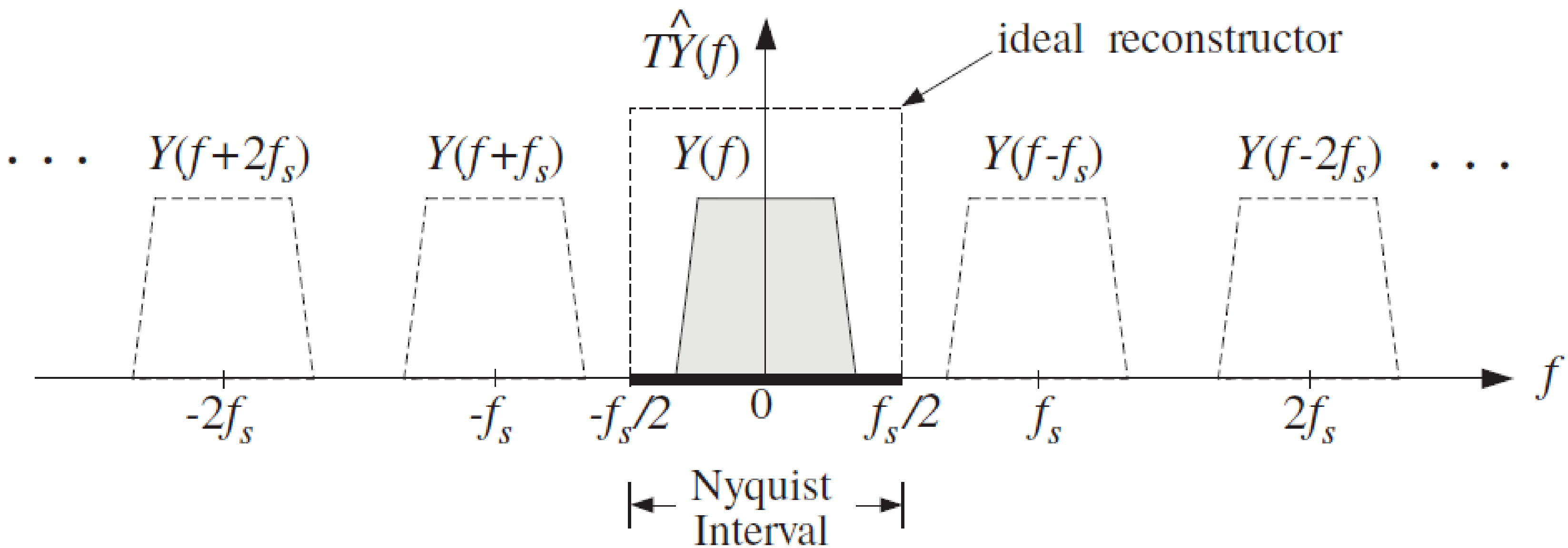
$$y_a(t) = \sum_{n=-\infty}^{\infty} y(nT)\,h(t-nT)$$

$$Y_a(f) = H(f)\hat{Y}(f) = H(f)\cdot\frac{1}{T}\sum_{m=-\infty}^{\infty} Y(f-mf_s)$$

Analog Reconstructors – Ideal

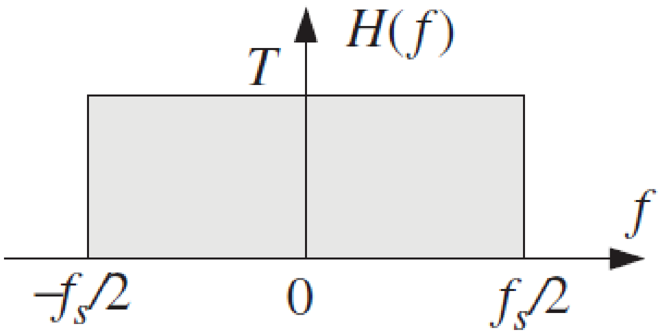
$$H(f) = \begin{cases} T, & \text{if } -\frac{f_s}{2} \le f \le \frac{f_s}{2} \\ 0, & \text{otherwise} \end{cases}$$





Analog Reconstructors – Ideal

$$H(f) = \begin{cases} T\,, & \text{if } -\dfrac{f_s}{2} \le f \le \dfrac{f_s}{2} \\ 0\,, & \text{otherwise} \end{cases}$$



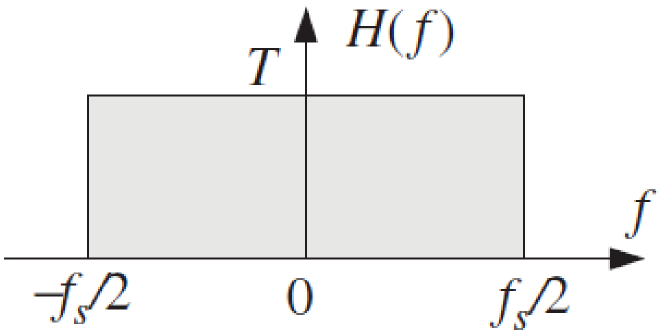
$$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{2\pi j f t}\, df = \int_{-f_s/2}^{f_s/2} T\, e^{2\pi j f t}\, df$$

$$h(t) = \frac{\sin(\pi f_s t)}{\pi f_s t} = \frac{\sin(\Omega_c t)}{\Omega_c t}$$

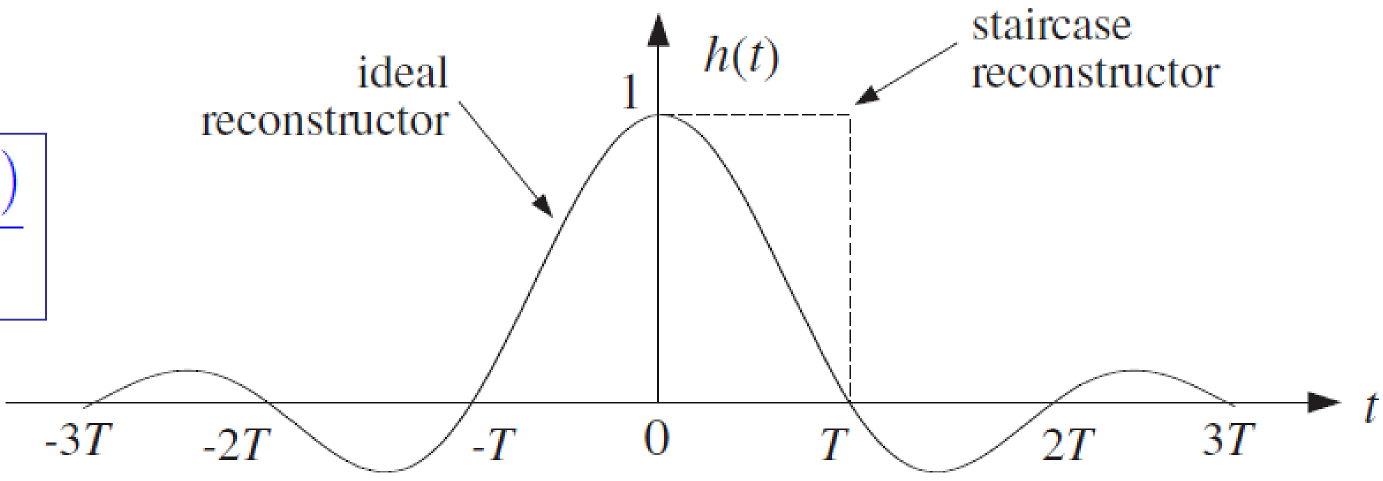
$$\Omega_c = \pi f_s = 2\pi \left(\frac{f_s}{2} \right) = \text{cutoff frequency in rads/sec}$$

Analog Reconstructors – Ideal

$$H(f) = \begin{cases} T\,, & \text{if } -\dfrac{f_s}{2} \le f \le \dfrac{f_s}{2} \\ 0\,, & \text{otherwise} \end{cases}$$

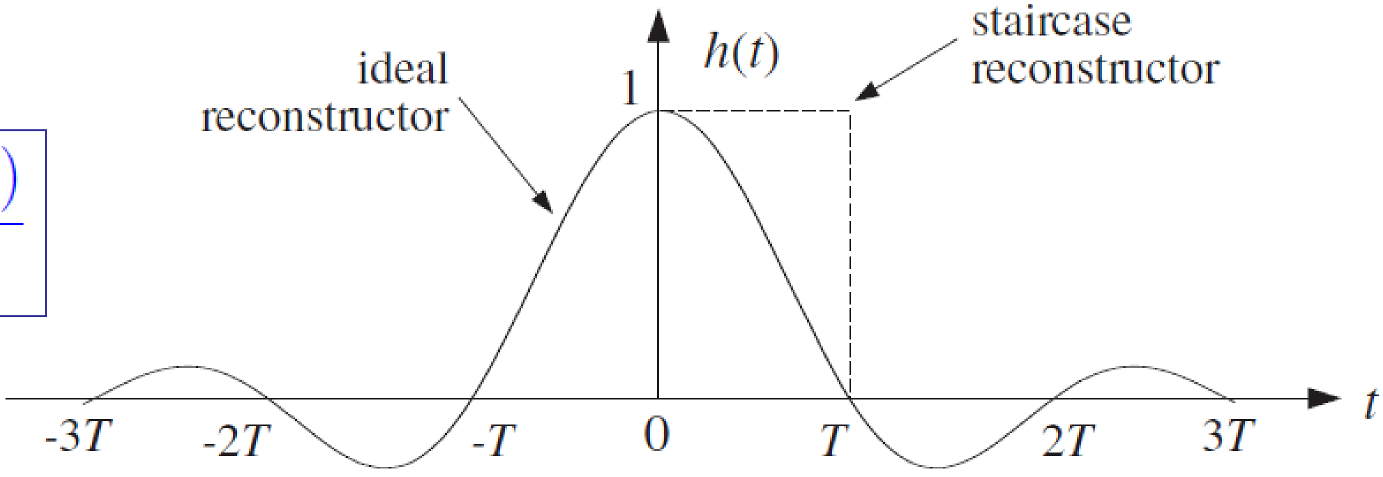


$$h(t) = \frac{\sin(\pi f_s t)}{\pi f_s t}$$



Analog Reconstructors – Ideal

$$h(t) = \frac{\sin(\pi f_s t)}{\pi f_s t}$$



$$y(t) = \sum_{n=-\infty}^{\infty} y(nT)\, h(t - nT) = \sum_{n=-\infty}^{\infty} y(nT)\, \frac{\sin\big(\pi f_s (t - nT)\big)}{\pi f_s (t - nT)}$$

$$Y_a(f) = H(f)\hat{Y}(f) = H(f) \cdot \frac{1}{T} \sum_{m=-\infty}^{\infty} Y(f - m f_s) = T \cdot \frac{1}{T} Y(f) = Y(f)$$

Sampling Theorem – Bandlimited Functions

$$y(t) = \sum_{n=-\infty}^{\infty} y(nT)\,\frac{\sin\bigl(\pi f_s(t-nT)\bigr)}{\pi f_s(t-nT)} = \sum_{n=-\infty}^{\infty} y(nT)\,f_n(t)$$

$$f_n(t) = \frac{\sin\bigl(\pi f_s(t-nT)\bigr)}{\pi f_s(t-nT)} = \text{orthogonal basis functions}$$

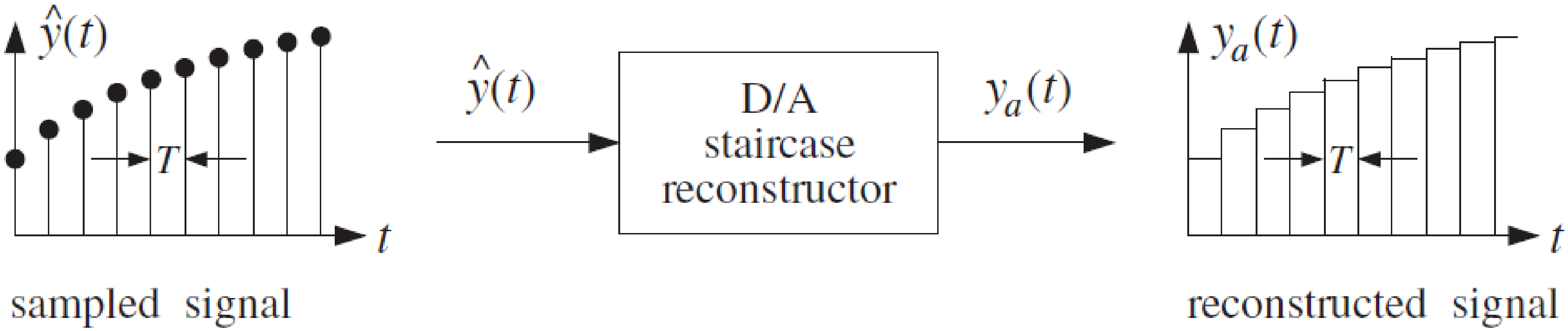
$$\int_{-\infty}^{\infty} f_n(t)\,f_m(t)\,dt = T\,\delta_{nm}$$

other orthonormal bases:

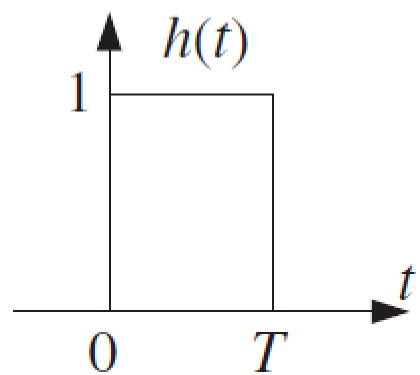
(1) prolate spheroidal wave functions

(2) spherical Bessel functions

Analog Reconstructors – Staircase



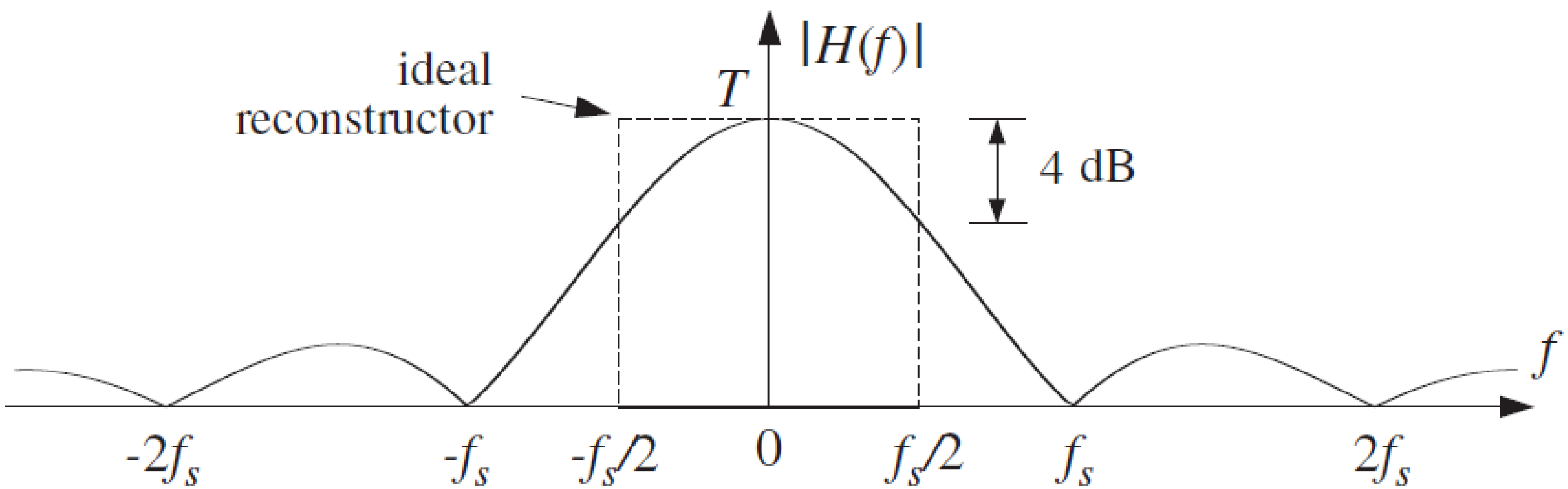
$$h(t) = u(t) - u(t-T) = \begin{cases} 1\,, & \text{if } \; 0 \le t \le T \\ 0\,, & \text{otherwise} \end{cases}$$



Analog Reconstructors – Staircase

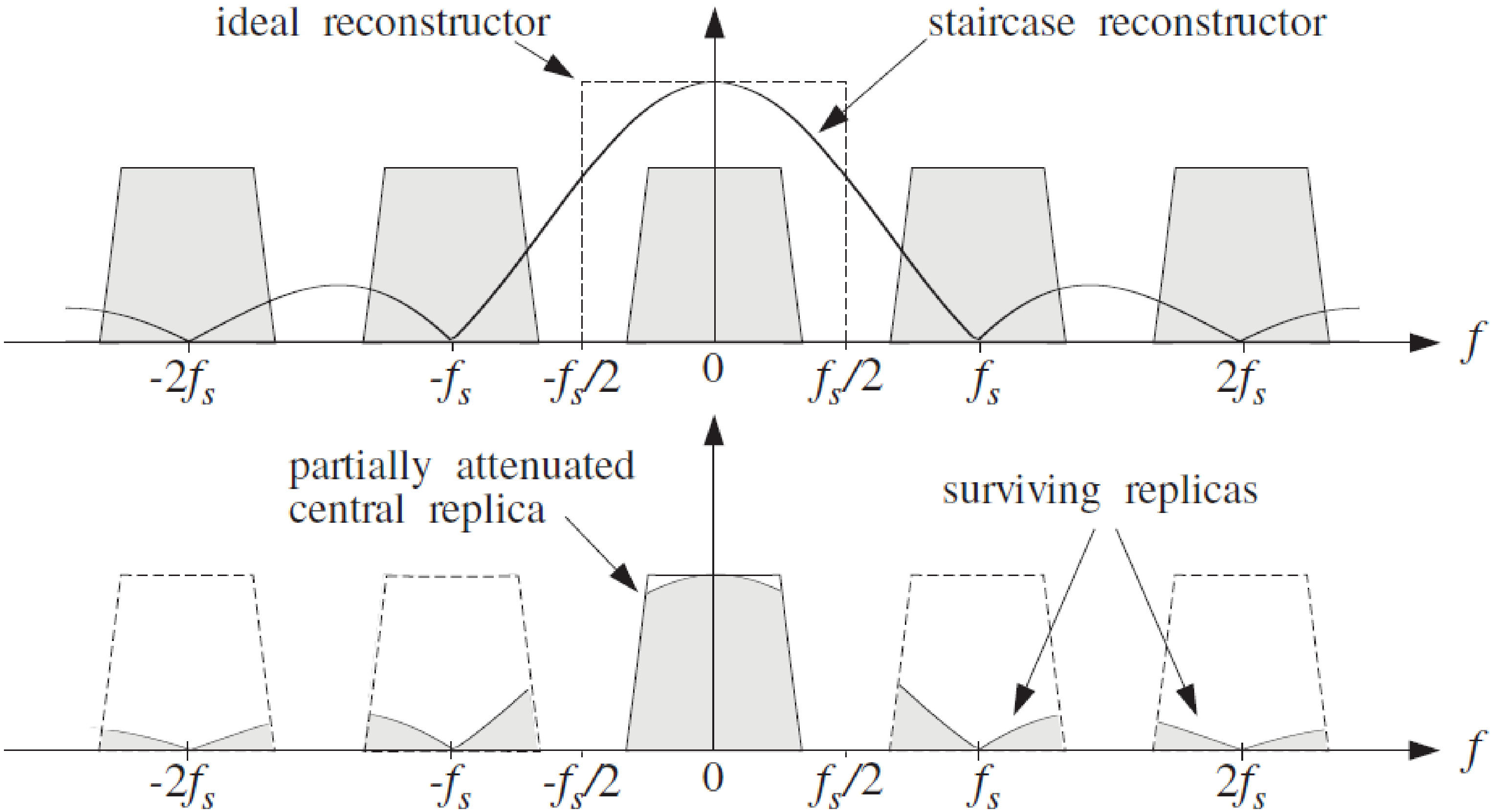
$$H(f) = H(s)\Big|_{s=2\pi jf} = \frac{1-e^{-sT}}{s}\Bigg|_{s=2\pi jf} = \frac{1-e^{-2\pi jfT}}{2\pi jf}$$

$$\boxed{H(f) = T\cdot\frac{\sin(\pi fT)}{\pi fT}\,e^{-\pi jfT}}$$

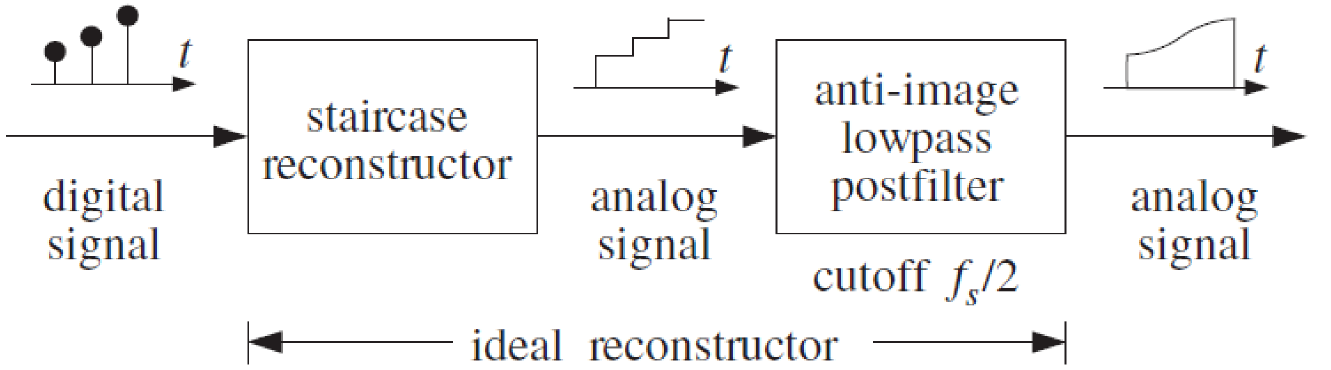


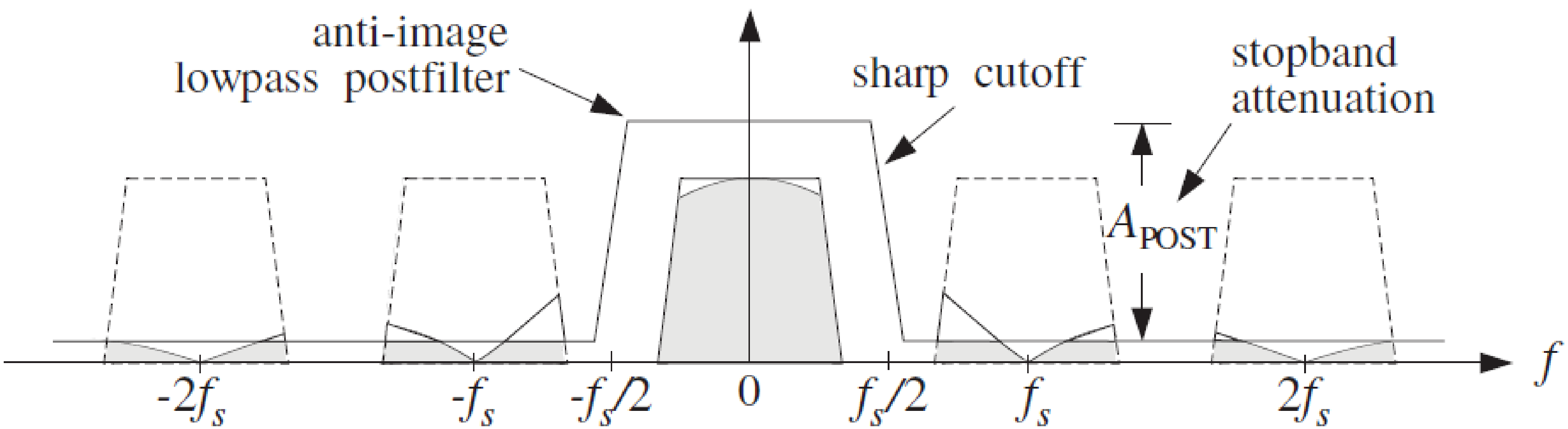
Analog Reconstructors – Staircase

$$H(f) = T \cdot \frac{\sin(\pi f T)}{\pi f T} e^{-\pi j f T}$$



Anti-Image Postfilters

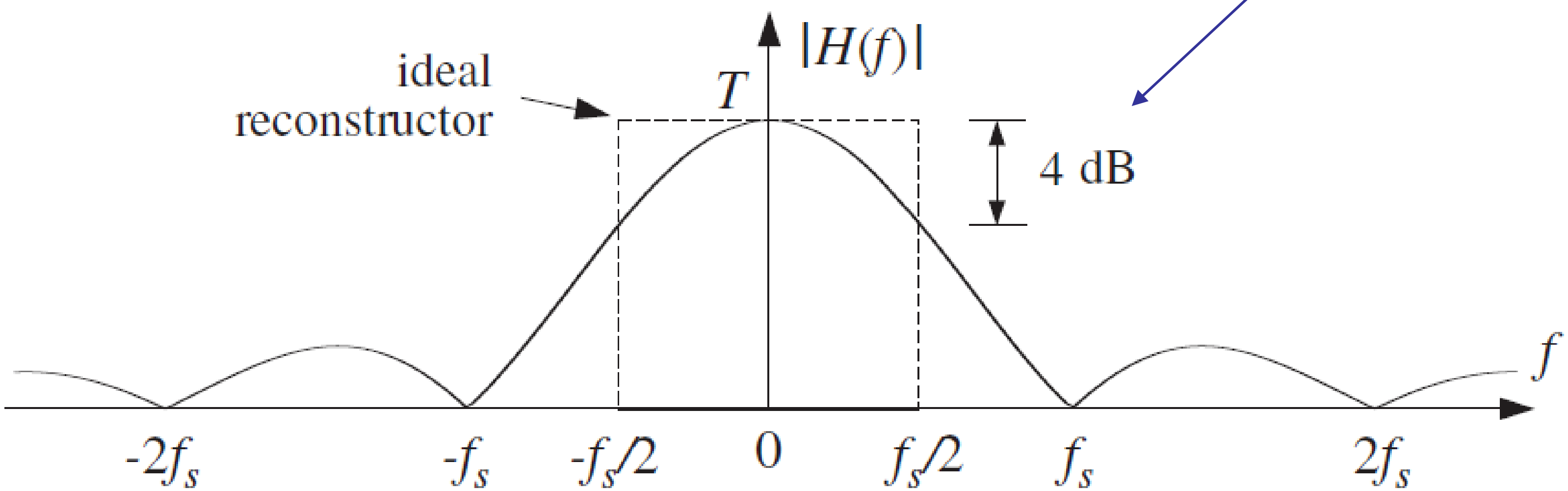




DAC Equalizer Filter

$$H(f) = T \cdot \frac{\sin(\pi f T)}{\pi f T} e^{-\pi j f T}$$

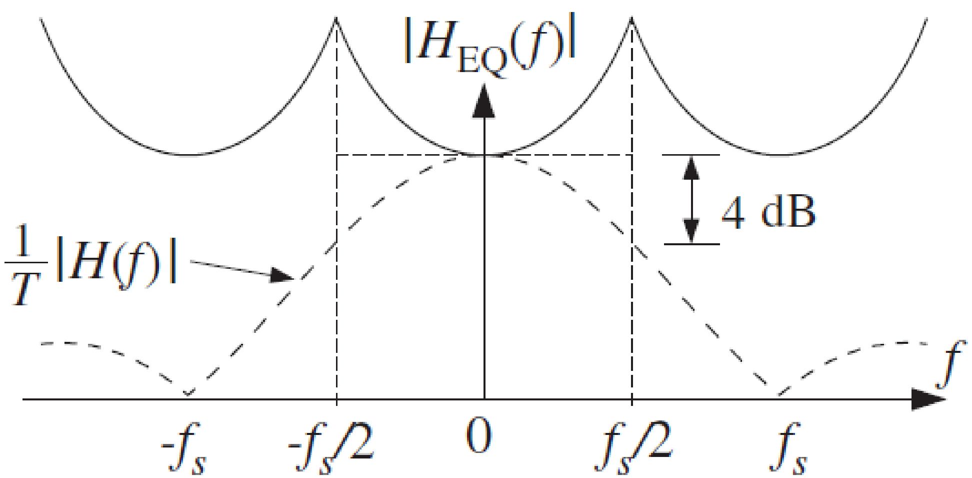
$$-20 \log_{10} \left| \frac{H(f_s/2)}{H(0)} \right| = -20 \log_{10} \left| \frac{\sin(\pi/2)}{\pi/2} \right| = 3.92 \text{ dB}$$



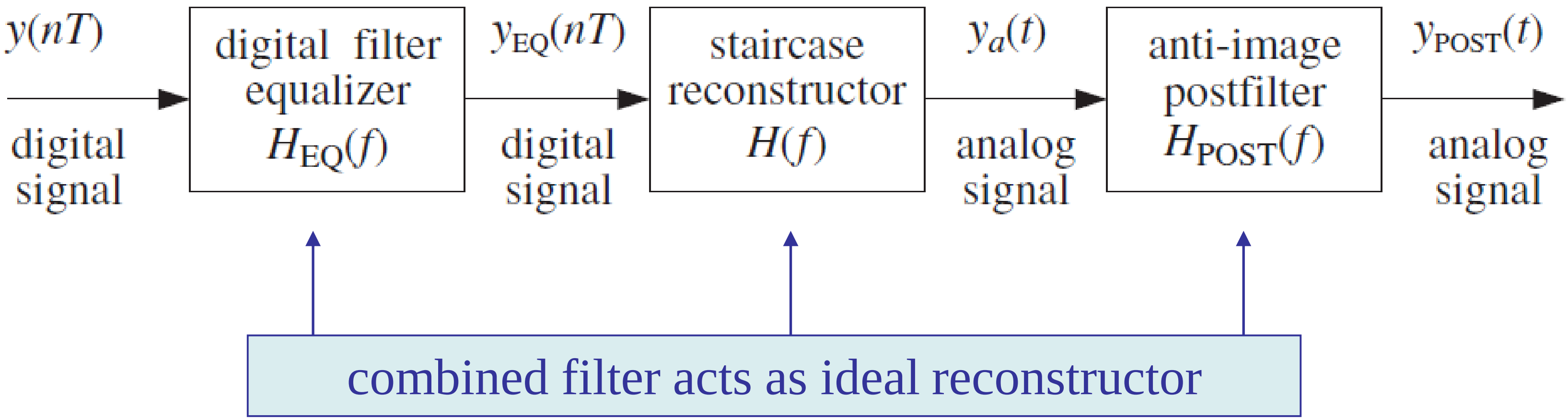
DAC Equalizer Filter

$$H(f) = T \cdot \frac{\sin(\pi f T)}{\pi f T} e^{-\pi j f T}$$

$$H_{\mathrm{EQ}}(f) = \frac{T}{H(f)} = \frac{\pi f T}{\sin(\pi f T)}, \quad |f| \leq \frac{f_s}{2}$$



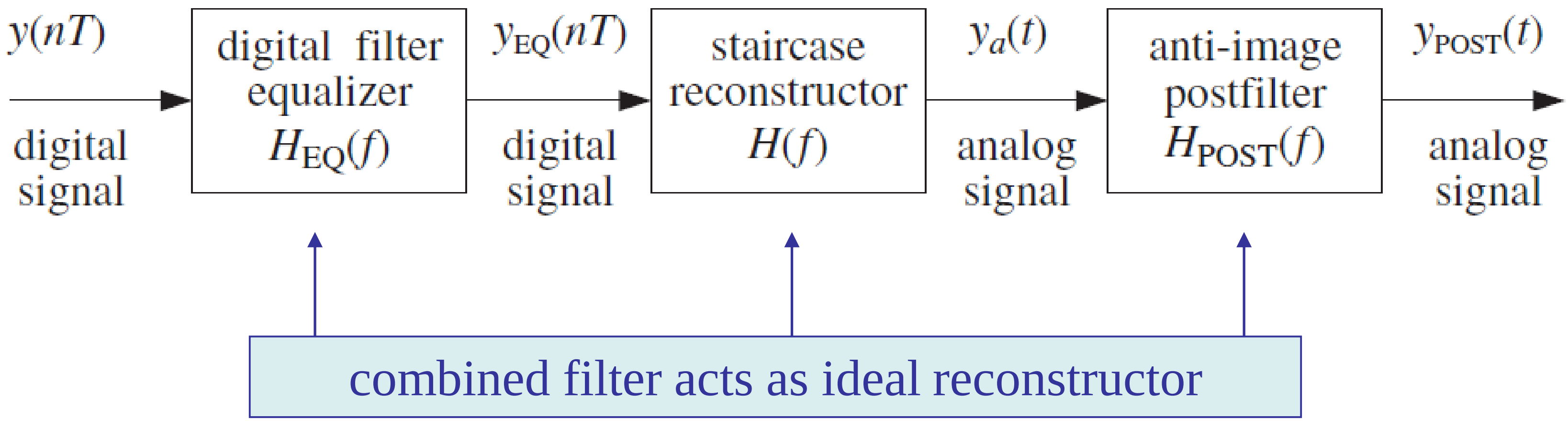
Equalizer – DAC – Postifilter Reconstructor



$$
\begin{aligned}
Y_{POST}(f) &= H_{POST}(f)\, Y_a(f) \\
&= H_{POST}(f)\, H(f)\, \hat{Y}_{EQ}(f) \\
&= H_{POST}(f)\, H(f)\, H_{EQ}(f)\, \hat{Y}(f)
\end{aligned}
$$

$$
\boxed{Y_{POST}(f) = H_{POST}(f)\, H(f)\, H_{EQ}(f)\, \hat{Y}(f) = 1 \cdot T \cdot \frac{1}{T} Y(f) = Y(f)}
$$

Equalizer – DAC – Postfilter Reconstructor – Example

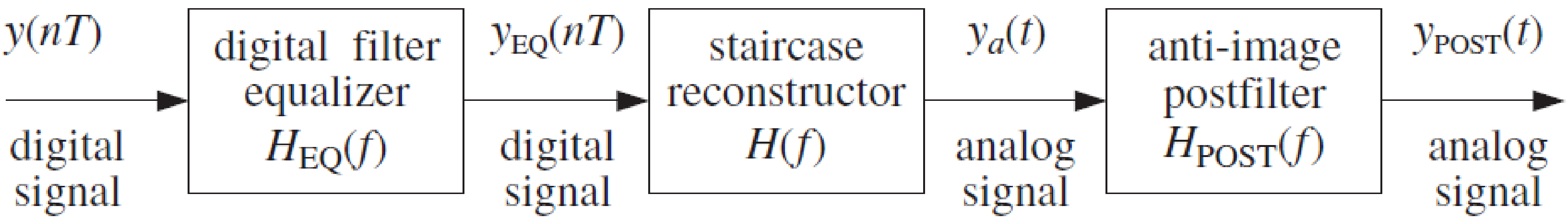


$$y(t) = e^{2\pi j f_0 t} \quad \Rightarrow \quad Y(f) = \delta(f - f_0)$$

$$\hat{y}(t) = \sum_{n=-\infty}^{\infty} y(nT)\delta(t - nT) = \sum_{n=-\infty}^{\infty} e^{2\pi j f_0 nT}\delta(t - nT)$$

$$\hat{Y}(f) = \frac{1}{T}\sum_{m=-\infty}^{\infty} Y(f - f_0 - mf_s) = \frac{1}{T}\sum_{m=-\infty}^{\infty} \delta(f - f_0 - mf_s)$$

Equalizer – DAC – Postifilter Reconstructor – Example



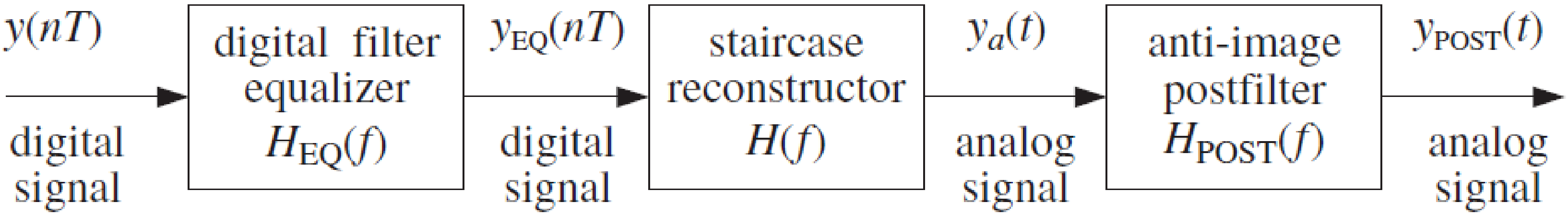
$$\hat{Y}(f) = \frac{1}{T}\sum_{m=-\infty}^{\infty} Y(f - f_0 - mf_s) = \frac{1}{T}\sum_{m=-\infty}^{\infty} \delta(f - f_0 - mf_s)$$

$$f_m = f_0 + mf_s$$

$$Y_a(f) = H_{EQ}(f)H(f)\hat{Y}(f) = \frac{1}{T}\sum_{m=-\infty}^{\infty} H_{EQ}(f_m)H(f_m)\delta(f - f_m)$$

$$Y_a(f) = \sum_{m=-\infty}^{\infty} \frac{H(f_m)}{H(f_0)}\delta(f - f_m)$$

Equalizer – DAC – Postfilter Reconstructor – Example

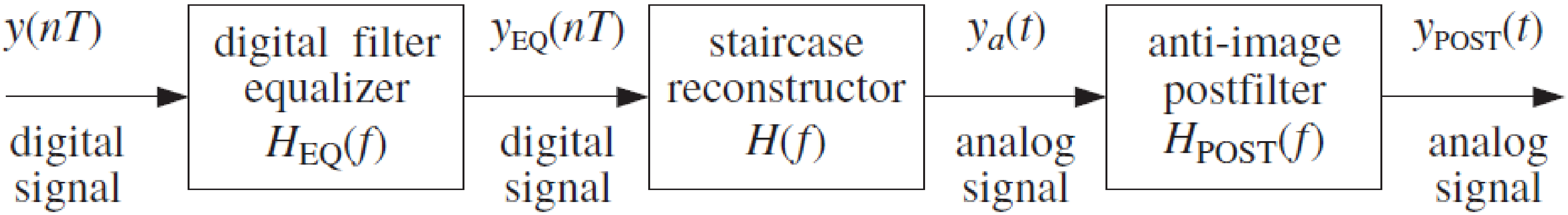


$$Y_a(f) = \sum_{m=-\infty}^{\infty} \frac{H(f_m)}{H(f_0)} \delta(f - f_m)$$

$$y_a(t) = \sum_{m=-\infty}^{\infty} \frac{H(f_m)}{H(f_0)} e^{2\pi j f_m t}$$

$$y_a(t) = \sum_{m=-\infty}^{\infty} \frac{f_0}{f_m} e^{2\pi j f_m t}$$

Equalizer – DAC – Postifilter Reconstructor – Example

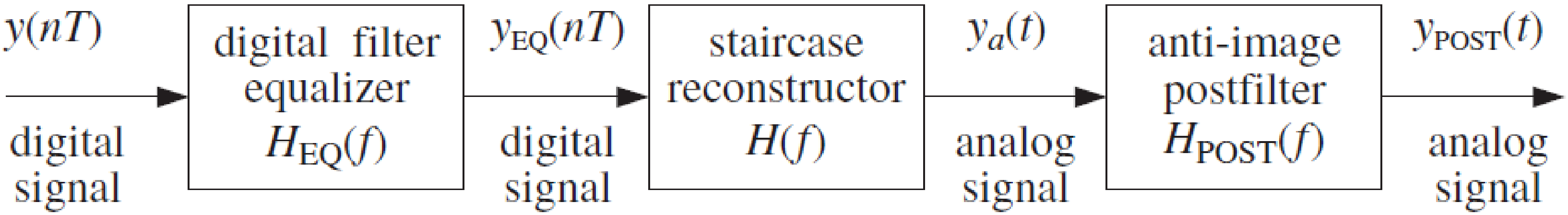


$$y_a(t) = \sum_{m=-\infty}^{\infty} \frac{f_0}{f_m}\, e^{2\pi j f_m t}$$

$$y_a(t) \approx \sum_{m=-M}^{M} w(m)\, \frac{f_0}{f_m}\, e^{2\pi j f_m t}$$

$$w(m) = 0.54 + 0.46 \cos\left(\frac{\pi m}{M}\right) = \text{Hamming window}$$

Equalizer – DAC – Postifilter Reconstructor – Example

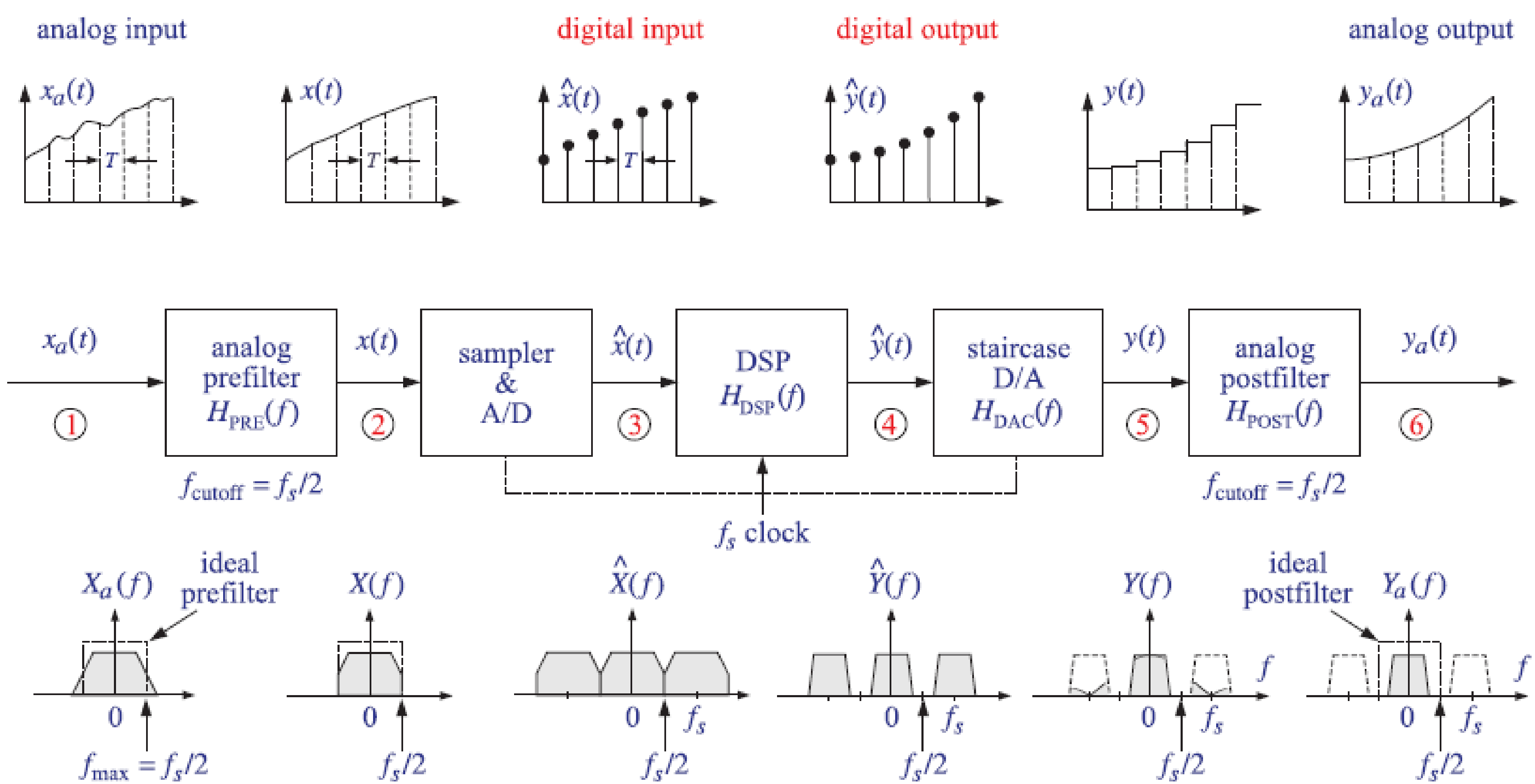



```
fc = fs/2;
N = 1;          % N = 2,3,4,5

[b,a] = butter(N, 2*pi*fc, 's');

% ya = ...

yfinal = lsim(tf(b,a),ya, t);

% see project-1 for more details
```

extract real parts

$$y_a(t) \approx \sum_{m=-M}^{M} w(m)\,\frac{f_0}{f_m}\,e^{2\pi j f_m t}$$

Basic Components of Conventional DSP System



Basic Components of Conventional DSP System

$$
\begin{aligned}
Y_a(f) &= H_{\mathrm{POST}}(f)\, Y(f) = H_{\mathrm{POST}}(f)\, H_{\mathrm{DAC}}(f)\, \hat{Y}(f) \\
&= H_{\mathrm{POST}}(f)\, H_{\mathrm{DAC}}(f)\, H_{\mathrm{DSP}}(f)\, \hat{X}(f) \\
&= H_{\mathrm{POST}}(f)\, H_{\mathrm{DAC}}(f)\, H_{\mathrm{DSP}}(f)\, \frac{1}{T}\left[X(f) + \text{replicas}\right] \\
&= H_{\mathrm{POST}}(f)\, H_{\mathrm{DAC}}(f)\, H_{\mathrm{DSP}}(f)\, \frac{1}{T}\left[H_{\mathrm{PRE}}(f)\, X_a(f) + \text{replicas}\right]
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
H_{\mathrm{POST}}(f)\, H_{\mathrm{DAC}}(f) &\simeq T \\
\text{replicas} &\simeq 0 \\
H_{\mathrm{PRE}}(f) &\simeq 1
\end{aligned}
}
$$

$$
\boxed{Y_a(f) = H_{\mathrm{DSP}}(f)\, X_a(f)}, \qquad \text{for} \quad |f| \le \frac{f_s}{2}
$$

Sampling Theorem – Bandlimited Functions

$$y(t) = \sum_{n=-\infty}^{\infty} y(nT)\,\frac{\sin\big(\pi f_s(t-nT)\big)}{\pi f_s(t-nT)} = \sum_{n=-\infty}^{\infty} y(nT)\, f_n(t)$$

$$f_n(t) = \frac{\sin\big(\pi f_s(t-nT)\big)}{\pi f_s(t-nT)} = \text{orthogonal basis functions}$$

$$\int_{-\infty}^{\infty} f_n(t) f_m(t)\, dt = T\,\delta_{nm}$$

other orthonormal bases:

(1) prolate spheroidal wave functions

(2) spherical Bessel functions

prolate spheroidal wave functions (PSWF)

orthonormal bases in the space of square-integrable
bandlimited functions with bandwidth, $[-\omega_0, \omega_0]$

(1) sinc-basis, sampling theorem (Nyquist, Shannon, Whittaker, Kotelnikov)
(2) prolate spheroidal wave functions (Slepian, Pollak, Landau)
(3) spherical Bessel functions

$$f(t) = \int_{-\omega_0}^{\omega_0} \hat{f}(\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi}\,, \qquad \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

applications:
signal extrapolation, deconvolution, communication systems,
waveform design, antennas, diffraction-limited optical systems,
super-resolving apertures, super-directive antenna arrays,
super-oscillations, laser resonators, and acoustics,

for more details, see, EWA book - Ch. 20 & 26

prolate spheroidal wave functions (PSWF)

prolate spheroidal wave functions are orthonormal bases in the space of square-integrable bandlimited functions with bandwidth, $[-\omega_0, \omega_0]$ that are maximally concentrated over the time interval $[-t_0, t_0]$.

time-bandwidth product, $c = t_0\, \omega_0$
Shannon number, $N_c = 2\, c / \pi$

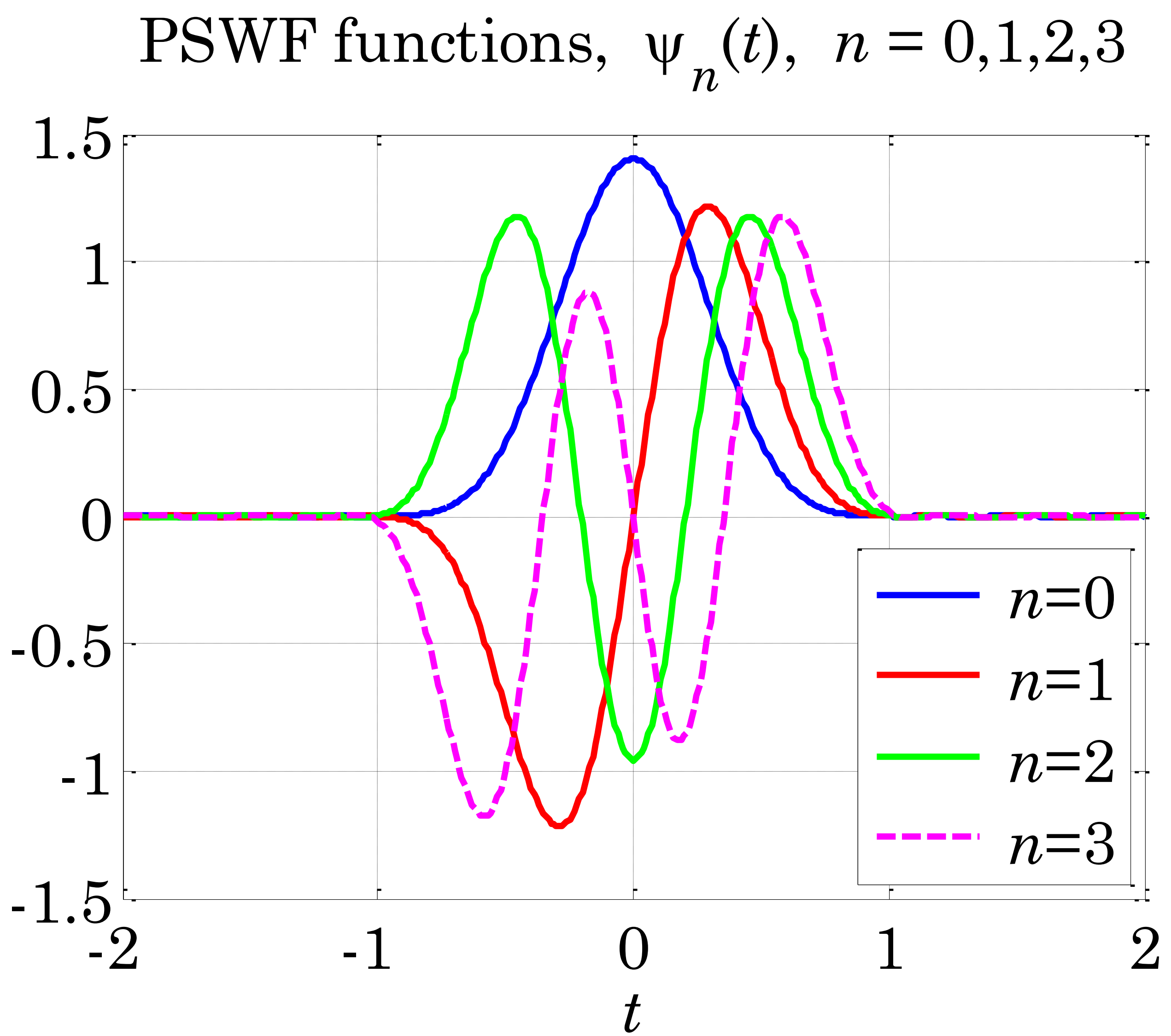
$$\psi_n(t) = \sqrt{\frac{\lambda_n}{t_0}} \sum_k \beta_{nk} \sqrt{k + \tfrac{1}{2}}\, P_k\!\left(\frac{t}{t_0}\right) = \sqrt{\frac{c}{2\pi t_0}} \sum_k \beta_{nk} \sqrt{k + \tfrac{1}{2}}\; 2\, i^{k-n}\, j_k(\omega_0 t)$$

$$\boxed{\int_{-t_0}^{t_0} \frac{\sin(\omega_0 (t - t'))}{\pi (t - t')}\, \psi_n(t')\, dt' = \lambda_n\, \psi_n(t)} \qquad n = 0, 1, 2, \ldots, \quad \text{for all } t$$

$$\mathcal{R}(f) = \frac{\int_{-t_0}^{t_0} f^2(t)\, dt}{\int_{-\infty}^{\infty} f^2(t)\, dt} = \max,\ \lambda_n$$

$$j_k(x) = \sqrt{\frac{\pi}{2x}}\, J_{k+\frac{1}{2}}(x)$$

prolate spheroidal wave functions (PSWF)



$t_0 = 1$

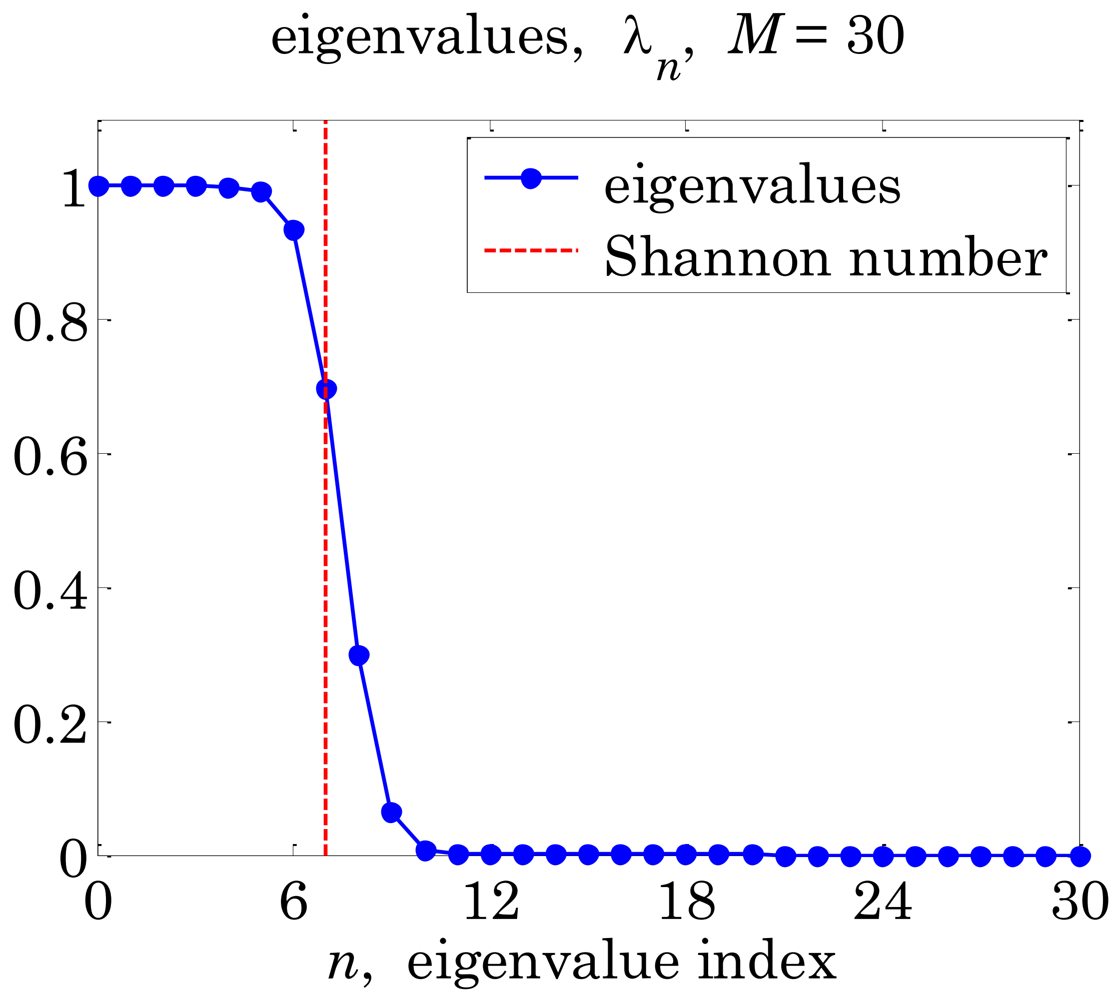
$f_0 = 2$

$\omega_0 = 2\pi f_0 = 4\pi$

$c = t_0\,\omega_0 = 4\pi$

$N_c = 2c/\pi = 8$

prolate spheroidal wave functions (PSWF)



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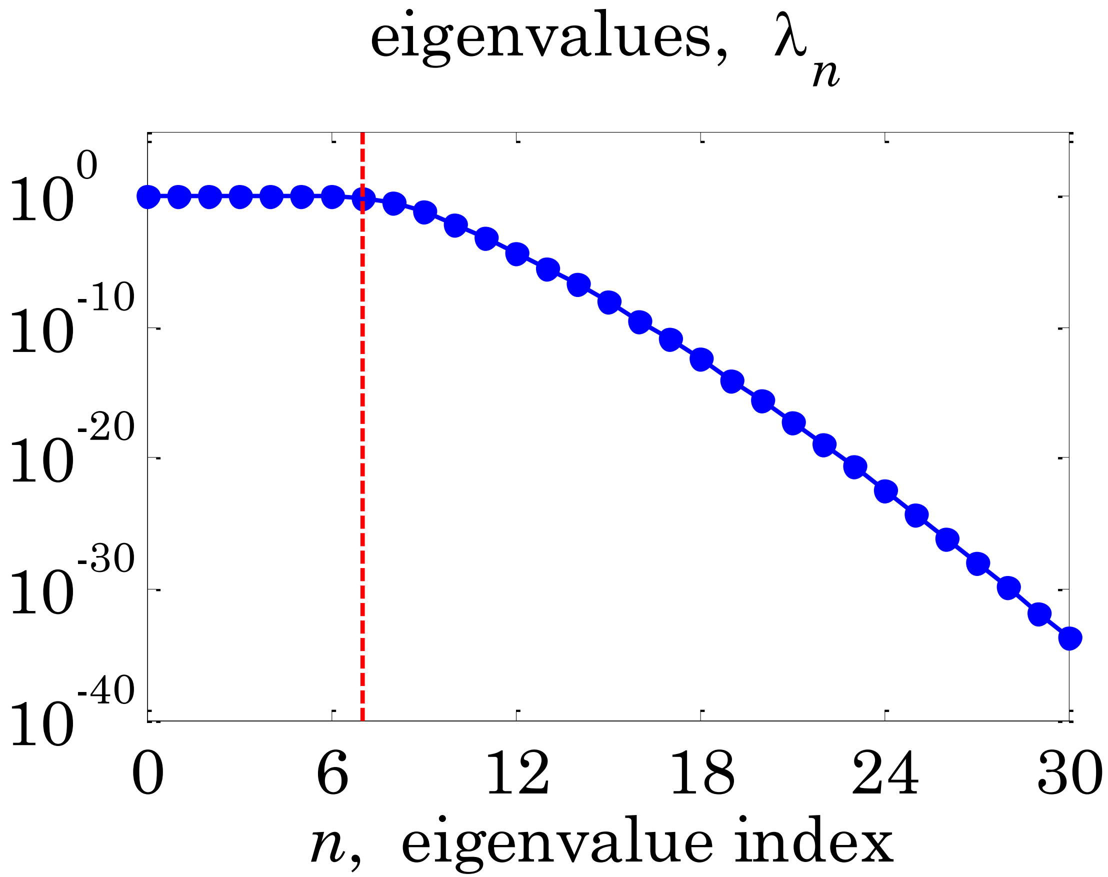

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$\omega_0 = 2\,\pi\, f_0 = 4\,\pi$

$c = t_0\, \omega_0 = 4\,\pi$

$N_c = 2\,c\,/\,\pi = 8$

Bandlimited Function Bases

$$\mu_n = i^n \left|\mu_n\right|, \quad \left|\mu_n\right| = \sqrt{\frac{2\pi\lambda_n}{c}}$$

$$f(t) = \sum_n c_n \psi_n(t)$$

$$\hat{f}(\omega) = \sum_n c_n \frac{2\pi}{\omega_0} \frac{1}{\mu_n} \psi_n\left(\frac{\omega t_0}{\omega_0}\right) \cdot \chi_{\omega_0}(\omega)$$

PSWF

$$f(t) = \sum_n c_n j_n(\omega_0 t)$$

$$\hat{f}(\omega) = \sum_n c_n \frac{\pi}{\omega_0 i^n} P_n\left(\frac{\omega}{\omega_0}\right) \cdot \chi_{\omega_0}(\omega)$$

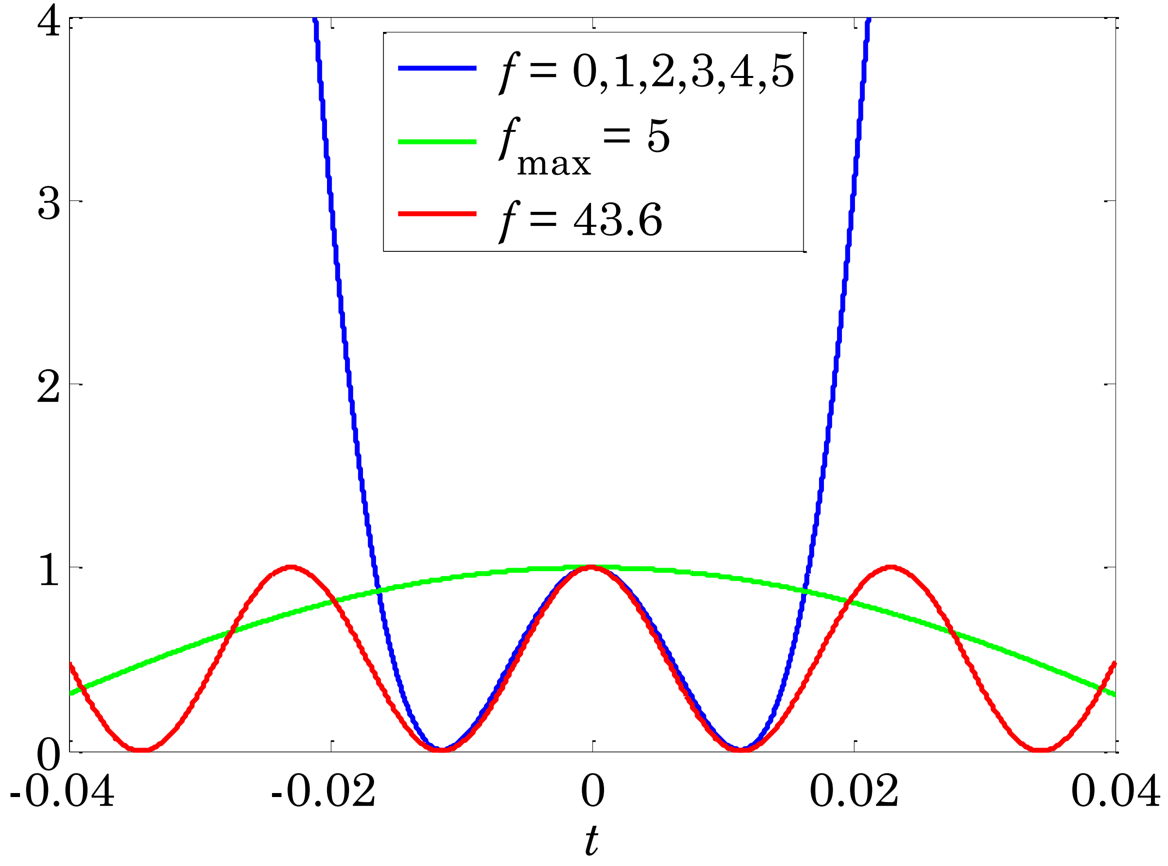
spherical Bessel

sinc

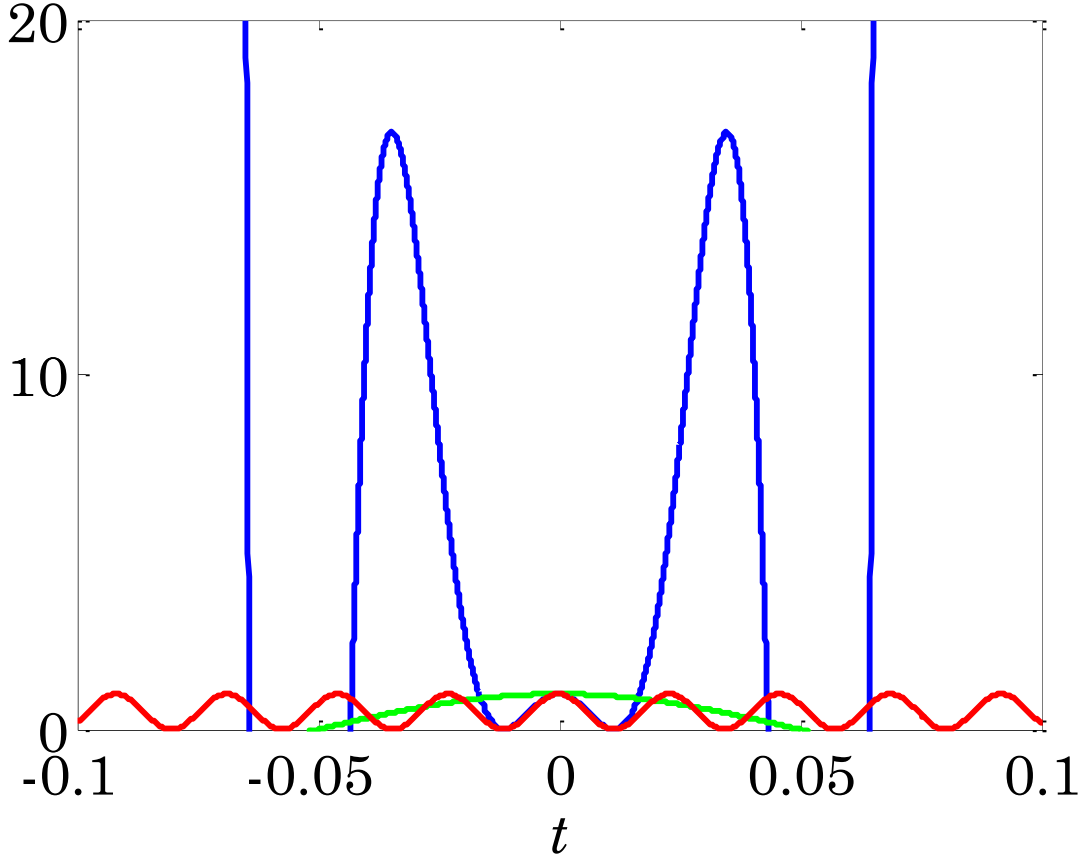
$$f(t) = \sum_n c_n \frac{\sin(\omega_0(t - nT_s))}{\pi(t - nT_s)} = T_s \sum_n f(nT_s) \frac{\sin(\omega_0(t - nT_s))}{\pi(t - nT_s)}$$

$$\hat{f}(\omega) = \sum_n c_n e^{-j\omega nT_s} \cdot \chi_{\omega_0}(\omega) = T_s \sum_n f(nT_s) e^{-j\omega nT_s} \cdot \chi_{\omega_0}(\omega)$$

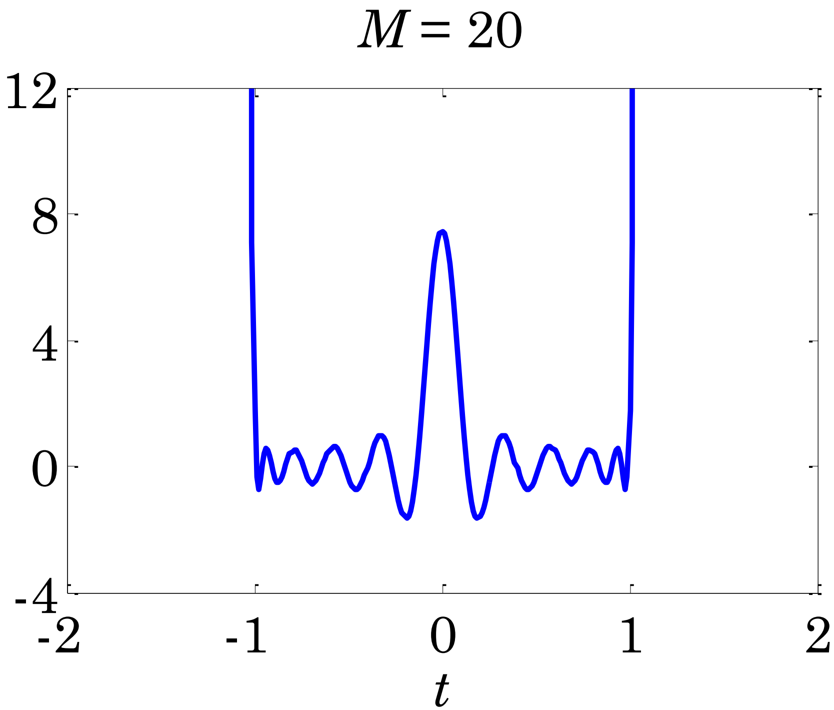
Super-Oscillation Examples

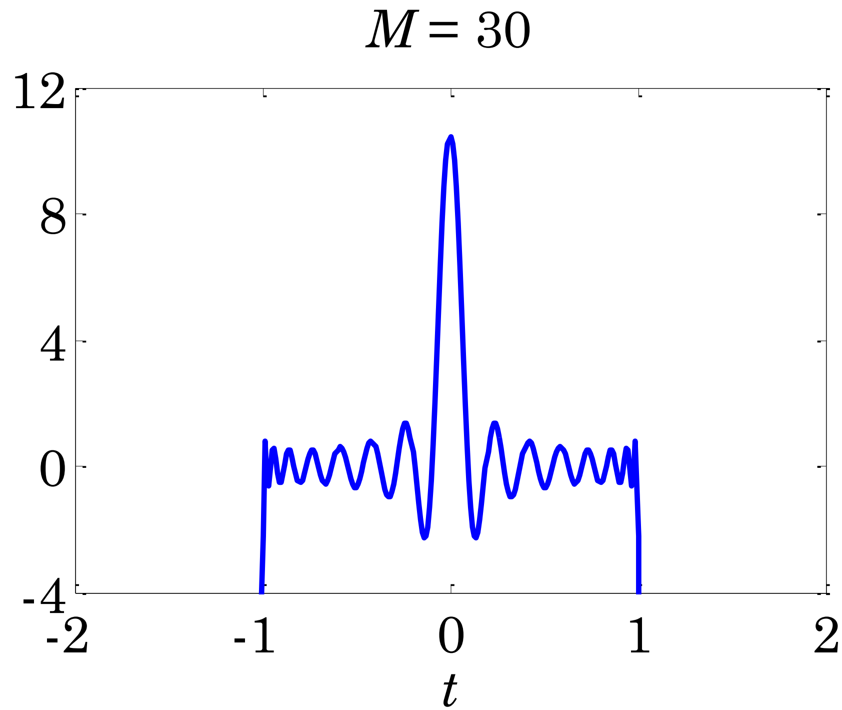


Super-Oscillation Examples

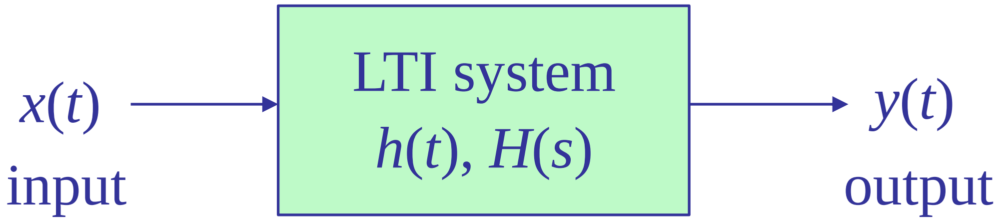


Super-Oscillation Examples





Linear Time-Invariant (LTI) Systems

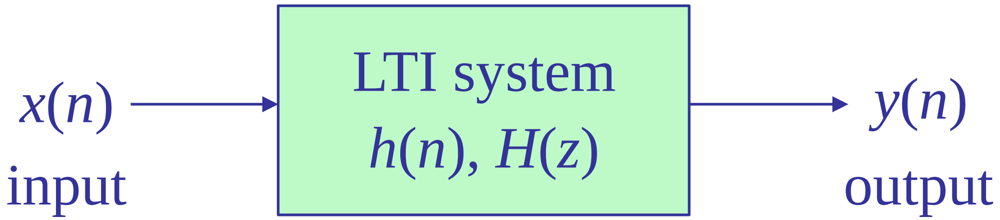


We will consider only a subset of LTI systems, namely, those described by linear constant-coefficient differential equations (LCCDEs) in the continuous-time case, or, by linear constant-coefficient difference equations in the discrete-time case.

In particular, the impulse response, $h(t)$, or $h[n]$, for this subset of systems is not an arbitrary function, but it satisfies an LCCDE, driven by an impulse.

Linear Time-Invariant (LTI) Systems

I2SP – Ch.3
O&S – Ch.2



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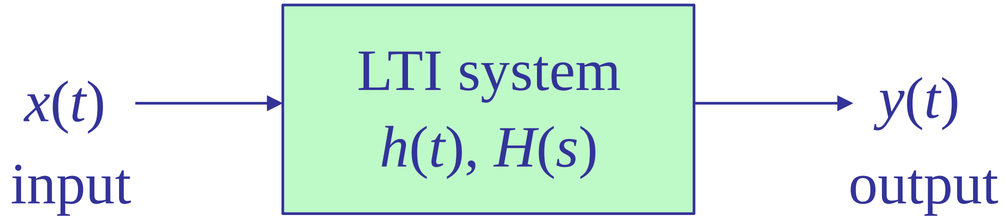
Linear Time-Invariant (LTI) Systems



LTI systems are convolvers

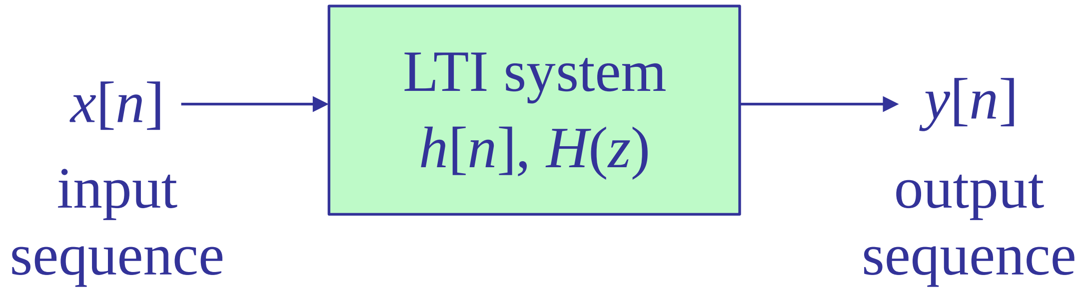
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

direct form

LTI form

Caveat: the convolutional output $y(t)$ assumes zero initial conditions and is referred to as the zero-state output – this will be clarified later.

Discrete-Time LTI Systems



LTI systems are convolvers

$$\boxed{y(n) = \sum_{m=-\infty}^{\infty} h(m)\,x(n-m) = \sum_{m=-\infty}^{\infty} x(m)\,h(n-m)}$$

direct form LTI form

Causality

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{0} h(\tau)x(t-\tau)\,d\tau + \int_{0}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

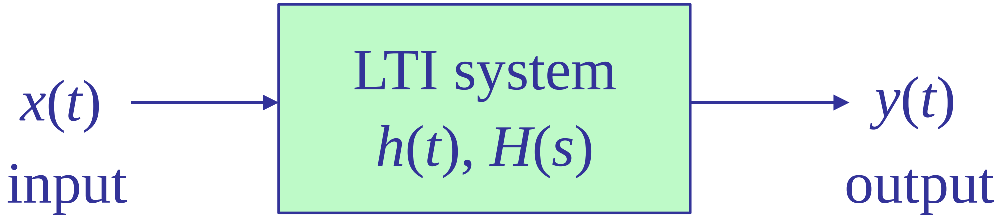
$$= \underbrace{\int_{0}^{\infty} h(-\tau)x(t+\tau)\,d\tau}_{\text{anti-causal part}} + \underbrace{\int_{0}^{\infty} h(\tau)x(t-\tau)\,d\tau}_{\text{causal part}}$$

for causal systems, $h(t) = 0$, for $t < 0$,

$$y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

system memory: present and past values of $x(t)$

Continuous-Time LTI Systems

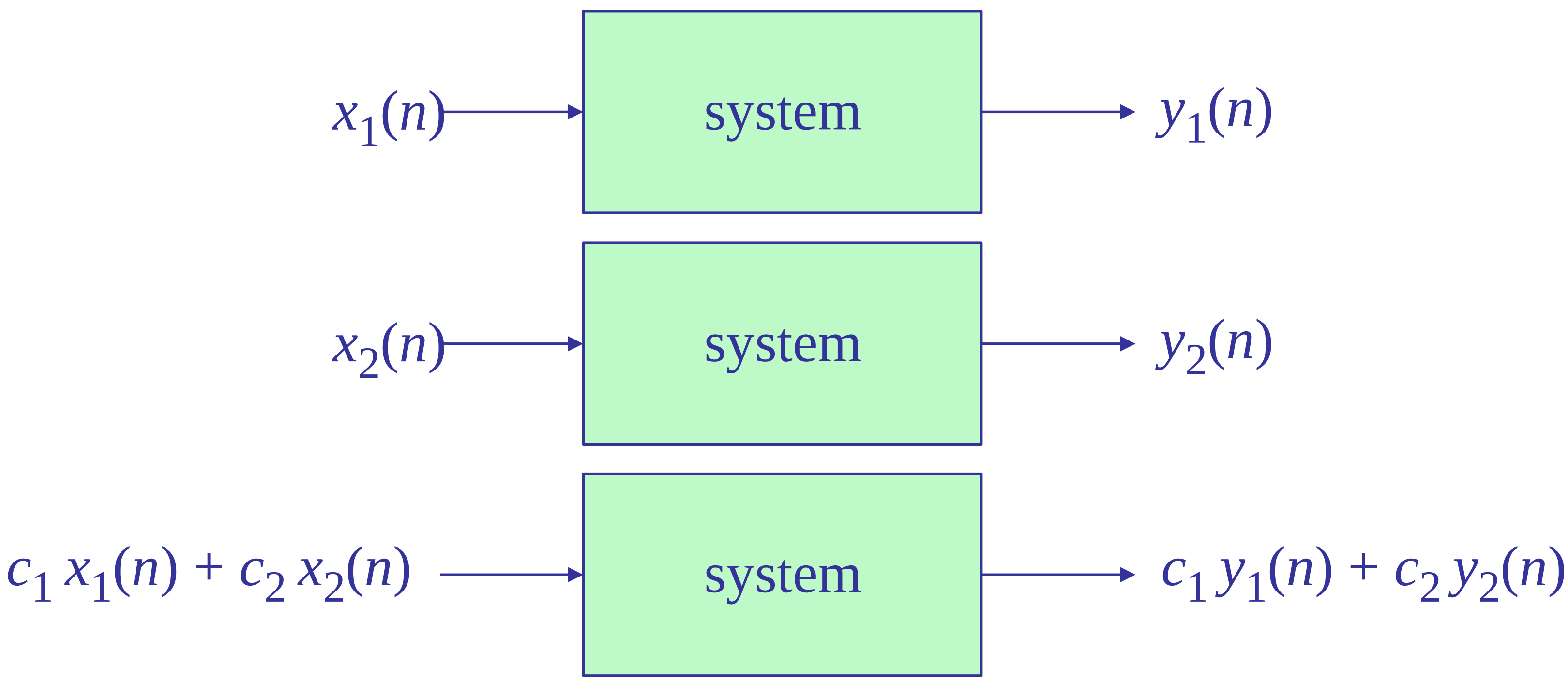


$$H(s) = \int_{0^-}^{\infty} h(t)\, e^{-st}\, dt = \text{Laplace transform}$$

$$H(\omega) = H(s)\Big|_{s=j\omega} = \int_{0^-}^{\infty} h(t)\, e^{-j\omega t}\, dt = \text{Fourier transform}$$

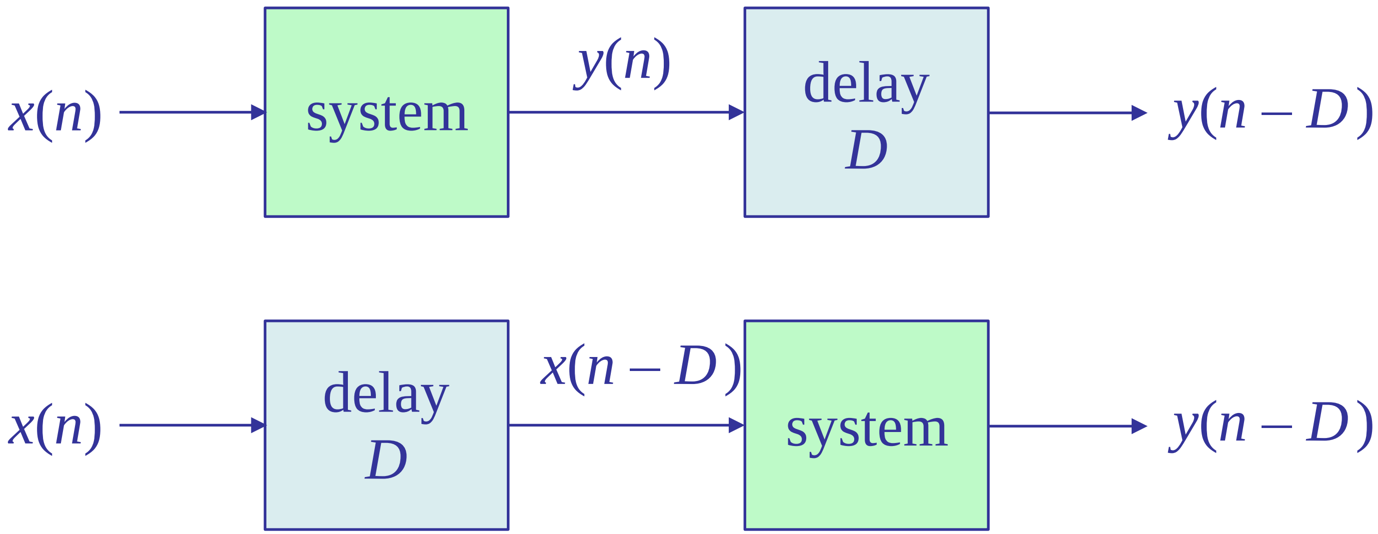
$$h(t) = \int_{-\infty}^{\infty} H(\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi} = \text{inverse Fourier transform}$$

Linearity



$x_1(n)$
system
$y_1(n)$
$x_2(n)$
system
$y_2(n)$
$c_1 x_1(n) + c_2 x_2(n)$
system
$c_1 y_1(n) + c_2 y_2(n)$

Time-Invariance



Discrete-Time LTI Systems

I2SP – Ch.3
O&S – Ch.2

$x(n)$ input $\longrightarrow$ LTI system $h(n)$, $H(z)$ $\longrightarrow$ $y(n)$ output

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} = \text{transfer function, } z\text{-transform}$$

$$H(\omega) = H(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} = \text{frequency response, DTFT}$$

$$h(n) = \int_{-\pi}^{\pi} H(\omega) e^{j\omega n}\, \frac{d\omega}{2\pi} = \text{inverse DTFT}$$

$$\boxed{\omega = \frac{2\pi f}{f_s}\,, \qquad z = e^{j\omega}}$$

$$h(n) = \oint_{\text{u.c.}} H(z)\, z^n\, \frac{dz}{2\pi j z} = \text{contour integral}$$

Discrete-Time LTI Systems

Alternative Notations

$$H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$

O&S

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$

I2SP

Discrete-Time LTI Systems

$$H(\omega) = H_R(\omega) + jH_I(\omega) = \text{frequency response}$$

$$H(\omega) = |H(\omega)|e^{j\theta(\omega)} = \text{polar form}$$

$$|H(\omega)| = \text{magnitude-response}$$

$$\theta(\omega) = \text{phase-response}$$

$$n_{\text{ph}}(\omega) = -\theta(\omega) = \text{phase-delay}$$

$$n_{\text{gr}}(\omega) = -\frac{d\theta(\omega)}{d\omega} = \text{group-delay}$$

Discrete-Time LTI Systems

$$\text{if } h(n) = \text{real-valued,}$$

$$H(-\omega) = H(\omega)^* = \text{Hermitian property}$$

$$H_R(-\omega) = H_R(\omega) = \text{even in } \omega$$

$$H_I(-\omega) = -H_I(\omega) = \text{odd in } \omega$$

$$|H(-\omega)| = |H(\omega)| = \text{even in } \omega$$

$$\theta(-\omega) = -\theta(\omega) = \text{odd in } \omega$$

$$n_{\text{ph}}(\omega) = \text{odd in } \omega$$

$$n_{\text{gr}}(\omega) = \text{even in } \omega$$

Stability

bounded-input bounded-output (BIBO) stability definition

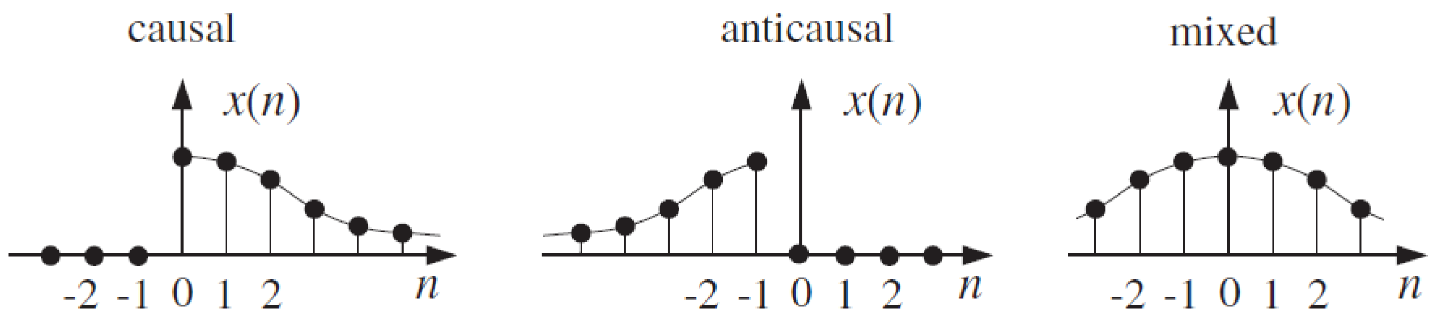
A system is BIBO-stable if every bounded input $x(n)$ results in a bounded output $y(n)$

necessary and sufficient condition for BIBO stability of LTI systems is that the inpulse response $h(n)$ be absolutely summable:

$$\boxed{\sum_{n=-\infty}^{\infty} |h(n)| < \infty}$$

for a proof, see I2SP, Problems 3.14 & 3.15

Stability vs. Causality



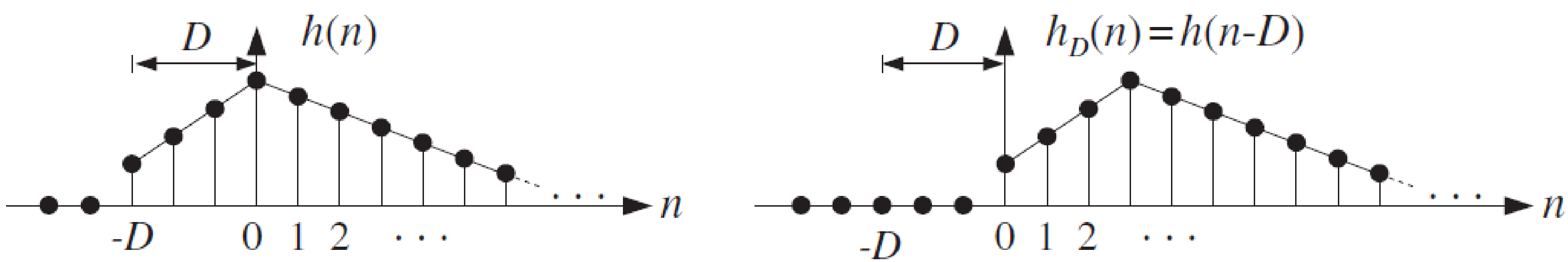
$$y_n = \sum_{m=-\infty}^{\infty} h_m x_{n-m}$$

$$y_n = \cdots + h_{-2}x_{n+2} + h_{-1}x_{n+1} + h_0 x_n + h_1 x_{n-1} + +h_2 x_{n+2} + \cdots$$

Stability vs. Causality

$$y_n = \sum_{m=-\infty}^{\infty} h_m x_{n-m} \approx \sum_{m=-D}^{\infty} h_m x_{n-m}$$

$$y_n \approx h_{-D} x_{n+D} + \cdots + h_{-1} x_{n+1} + h_0 x_n + h_1 x_{n-1} + + h_2 x_{n+2} + \cdots$$



finitely anticausal can be made causal by a delay

output comes out with the same delay, $y(n\text{-}D)$

which is usually OK, unless there is output feedback

Sinusoidal Response

Input complex sinusoid is assumed to be infinitely-long and double-sided (i.e., we are looking at the steady-state behavior of the system),

$$Ae^{j\omega n} \longrightarrow \boxed{H(\omega)} \longrightarrow H(\omega)Ae^{j\omega n}$$

Moreover, if the LTI system has real-valued impulse response $h(n)$, then the following results also hold for real-valued input sinusoids,

$$\mathrm{Re}\left[(Ae^{j\omega n}\right] \longrightarrow \boxed{H(\omega)} \longrightarrow \mathrm{Re}\left[H(\omega)Ae^{j\omega n}\right]$$

$$\mathrm{Im}\left[(Ae^{j\omega n}\right] \longrightarrow \boxed{H(\omega)} \longrightarrow \mathrm{Im}\left[H(\omega)Ae^{j\omega n}\right]$$

complex sinusoids can be viewed as the eigenfunctions of LTI systems

Filtering in the Frequency Domain

$$x(n) \longrightarrow \boxed{H(\omega)} \longrightarrow y(n)$$

$$x(n) = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

$$y(n) = \int_{-\pi}^{\pi} H(\omega)\, X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

i.e., filtering is spectral reshaping of the input spectrum

Filtering in the Frequency Domain - Example

$$x(n) \longrightarrow \boxed{H(\omega)} \longrightarrow y(n)$$

$$x(n) = A_1 e^{j\omega_1 n} + A_2 e^{j\omega_2 n}$$

$$X(\omega) = 2\pi A_1 \delta(\omega - \omega_1) + 2\pi A_2 \delta(\omega - \omega_2)$$

$$y(n) = \int_{-\pi}^{\pi} H(\omega) X(\omega) e^{j\omega n} \frac{d\omega}{2\pi}$$

$$= \int_{-\pi}^{\pi} H(\omega) \left[A_1 \delta(\omega - \omega_1) + A_2 \delta(\omega - \omega_2)\right] e^{j\omega n} \, d\omega$$

$$= H(\omega_1) A_1 e^{j\omega_1 n} + H(\omega_2) A_2 e^{j\omega_2 n}$$