# Delta Function Definitions

Examples of definitions of the Dirac delta function, $\delta(t)$, as a limit of ordinary functions:

$$\delta(t) = \lim_{\epsilon\to 0} f_\epsilon(t)$$

Since the Fourier transform of $\delta(t)$ is $\Delta(\omega) = 1$, one can look at the limit of either the time function $f_\epsilon(t)$ itself or its Fourier transform $F_\epsilon(\omega)$, that is, verify that,

$$\lim_{\epsilon\to 0} F_\epsilon(\omega) = \Delta(\omega) = 1, \quad \text{for all } \omega$$

where,

$$F_\epsilon(\omega) = \int_{-\infty}^{\infty} f_\epsilon(t)e^{-j\omega t}dt \quad \Leftrightarrow \quad f_\epsilon(t) = \int_{-\infty}^{\infty} F_\epsilon(\omega)e^{j\omega t}\frac{d\omega}{2\pi}$$

In all cases, we must have the normalization:

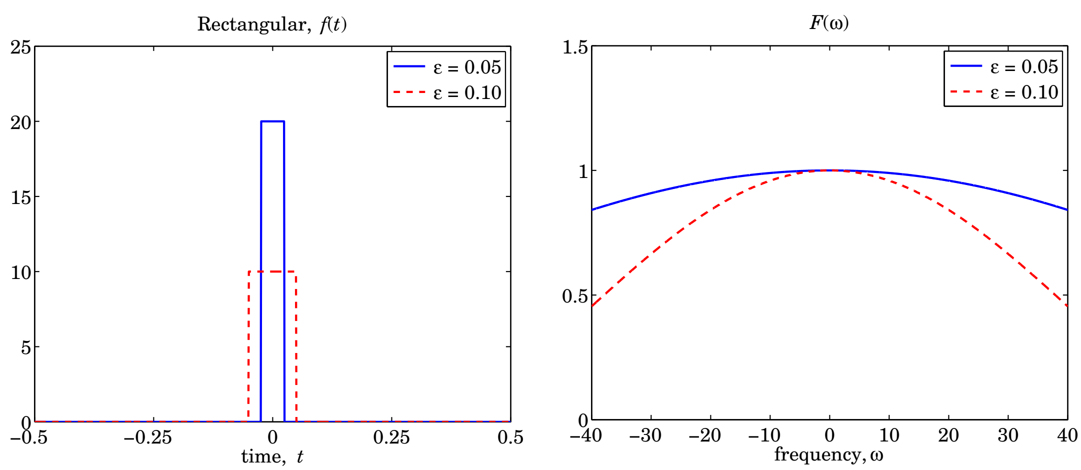
$$\int_{-\infty}^{\infty} f_\epsilon(t)dt = 1$$

or, equivalently, $F_\epsilon(0) = 1$, at DC, $\omega = 0$.

## 1. Rectangular

Here, $u(t)$ is the unit-step and $f_\epsilon(t)$ is a rectangular pulse of height $1/\epsilon$, over the interval, $-\frac{1}{2}\epsilon \le t \le \frac{1}{2}\epsilon$,

$$\boxed{f_\epsilon(t) = \frac{1}{\epsilon}\left[u\big(t+\tfrac{1}{2}\epsilon\big) - u\big(t-\tfrac{1}{2}\epsilon\big)\right] \quad \xrightarrow{FT} \quad F_\epsilon(\omega) = \frac{\sin(\omega\epsilon/2)}{\omega\epsilon/2}}$$

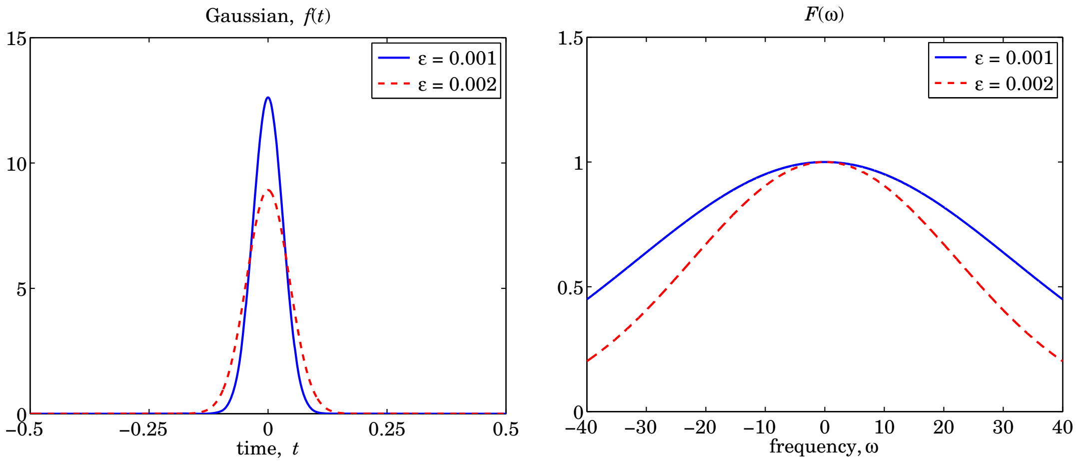



from the definition of derivatives, the limit $\epsilon \to 0$, also implies the usual property, $\delta(t) = \dot{u}(t)$,

$$\delta(t) = \lim_{\epsilon\to 0} f_\epsilon(t) = \lim_{\epsilon\to 0}\left[\frac{u\big(t+\frac{1}{2}\epsilon\big) - u\big(t-\frac{1}{2}\epsilon\big)}{\epsilon}\right] = \frac{du(t)}{dt}$$

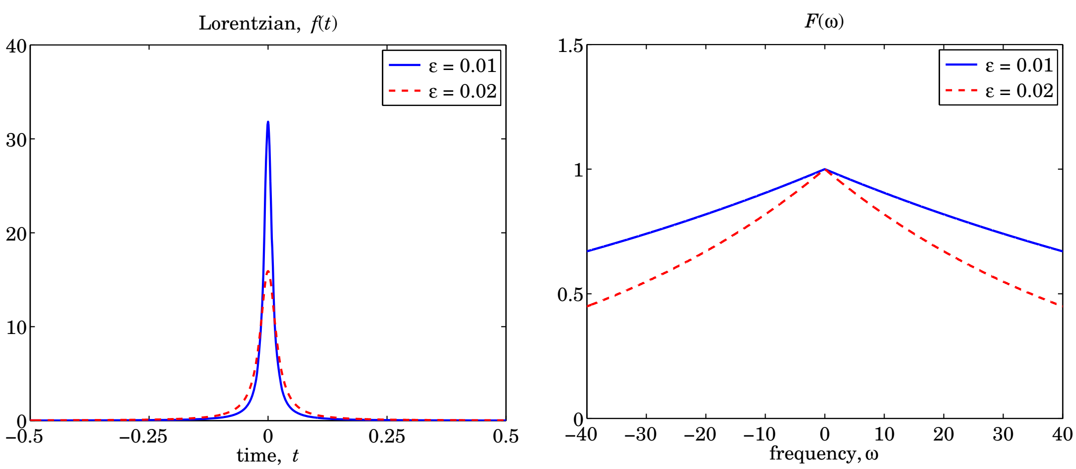
## 2. Gaussian

$$\boxed{f_\epsilon(t) = \frac{1}{\sqrt{2\pi\epsilon}}\, e^{-t^2/2\epsilon} \quad \xrightarrow{FT} \quad F_\epsilon(\omega) = e^{-\epsilon\omega^2/2}}$$




## 3. Lorentzian

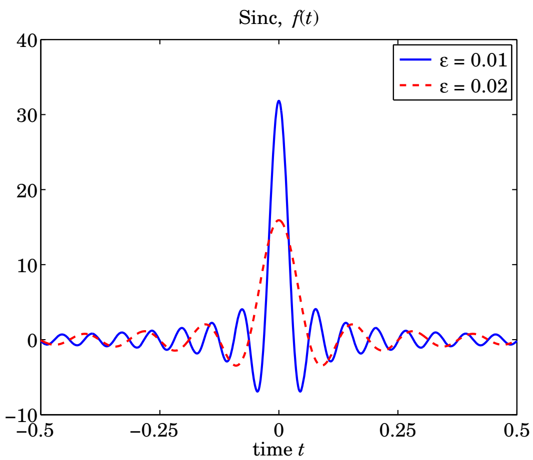
$$\boxed{f_\epsilon(t) = \frac{1}{\pi}\, \frac{\epsilon}{\epsilon^2 + t^2} \quad \xrightarrow{FT} \quad F_\epsilon(\omega) = e^{-\epsilon|\omega|}}$$

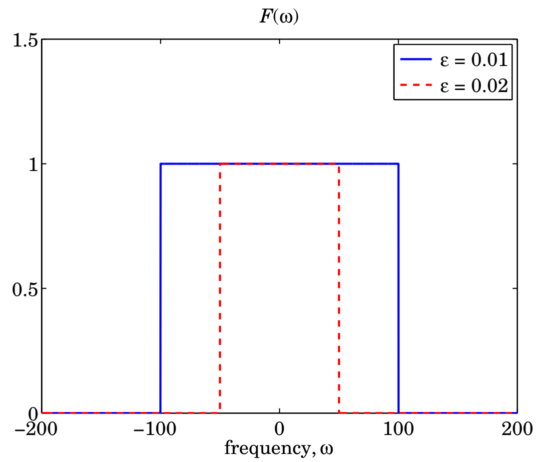


## 4. Sinc

$$\boxed{f_\epsilon(t) = \frac{\sin(t/\epsilon)}{\pi t} \quad \xrightarrow{FT} \quad F_\epsilon(\omega) = u_h\left(\frac{1}{\epsilon} - |\omega|\right) = \begin{cases} 1, & |\omega| < 1/\epsilon \\ 0.5, & |\omega| = 1/\epsilon \\ 0, & |\omega| > 1/\epsilon \end{cases}}$$

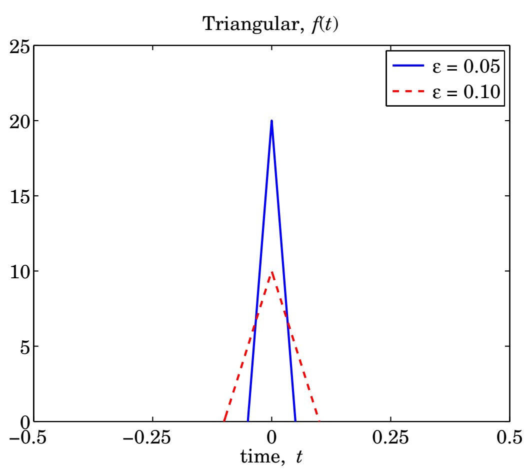
where $u_h(x)$ denotes the Heaviside unit-step function.

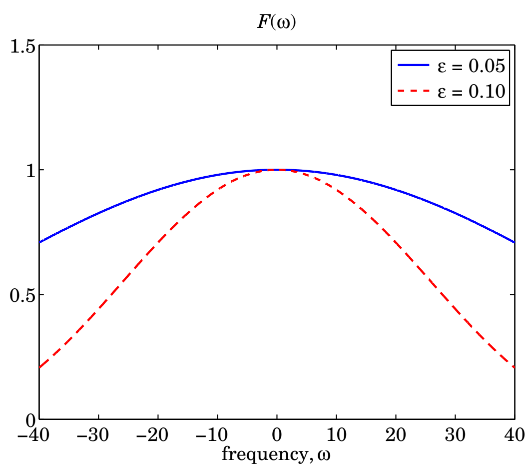



## 5. Triangular

$$\boxed{f_\epsilon(t) = \begin{cases} \dfrac{1}{\epsilon^2}\big(\epsilon - |t|\big)\,, & |t| \le \epsilon \\ 0\,, & |t| > \epsilon \end{cases} \quad \xrightarrow{FT} \quad F_\epsilon(\omega) = \frac{\sin^2(\omega\epsilon/2)}{(\omega\epsilon/2)^2}}$$



## 6. Delta Function Properties - Summary

$$\delta(t) = \delta(-t)$$

$$\delta(t) = |a|\delta(at)$$

$$f(t)\delta(t) = f(0)\delta(t)$$

$$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$$

$$\delta(t^2 - t_0^2) = \frac{1}{2|t_0|}\left[\delta(t - t_0) + \delta(t + t_0)\right]$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)\,, \quad \text{(sifting)}$$

$$\int_{t_0-\epsilon}^{t_0+\epsilon} f(t)\delta(t - t_0)dt = f(t_0)\,, \quad \epsilon > 0$$

$$\delta(t) = \int_{-\infty}^{\infty} e^{j\omega t}\,\frac{d\omega}{2\pi}\,, \quad \text{(inverse FT)}$$

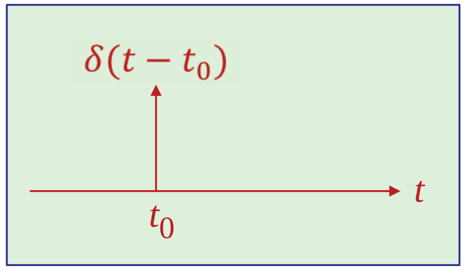
$$\varDelta(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{j\omega t}\,dt = 1\,, \quad \forall\,\omega\,, \quad \text{(FT)}$$

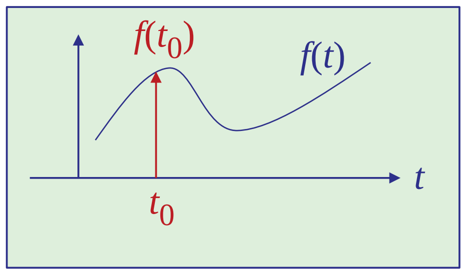
$$\dot{\delta}(-t) = -\dot{\delta}(t)$$

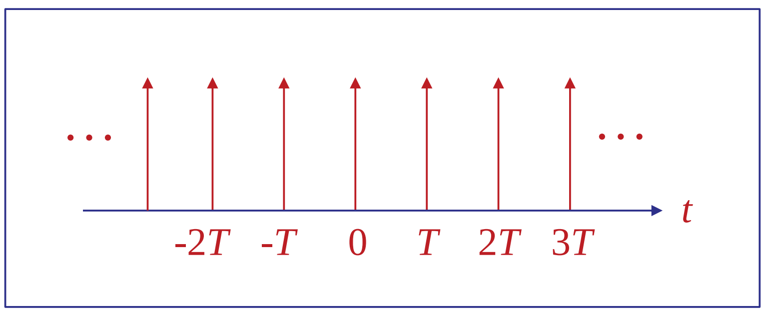
$$t\dot{\delta}(t) = -\delta(t)$$

$$\int_{-\infty}^{\infty} f(t)\dot{\delta}(t - t_0)dt = -\dot{f}(t_0)\,, \quad \text{(sifting)}$$

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{2\pi jkt/T}\,, \quad \text{(Dirac comb)}$$


δ(t − t0)
t
t0


f(t0)
f(t)
t
t0


. . .
. . .
t
-2T -T 0 T 2T 3T